

Lecture 4

October 22, 2002

◆ Transmission Line Review

◆ Equations, Definitions, Processes

◆ Reflection and Transmissions Coefficients

■ Coaxial Line

▣ Arbitrary Impedance

▣ Arbitrary Admittance

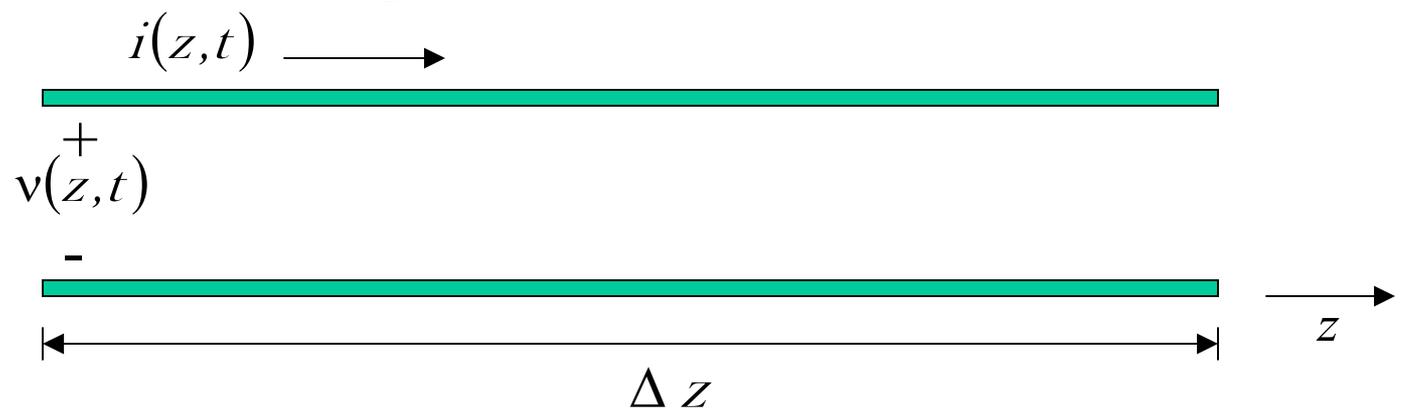
▣ Short Circuit

▣ Open Circuit

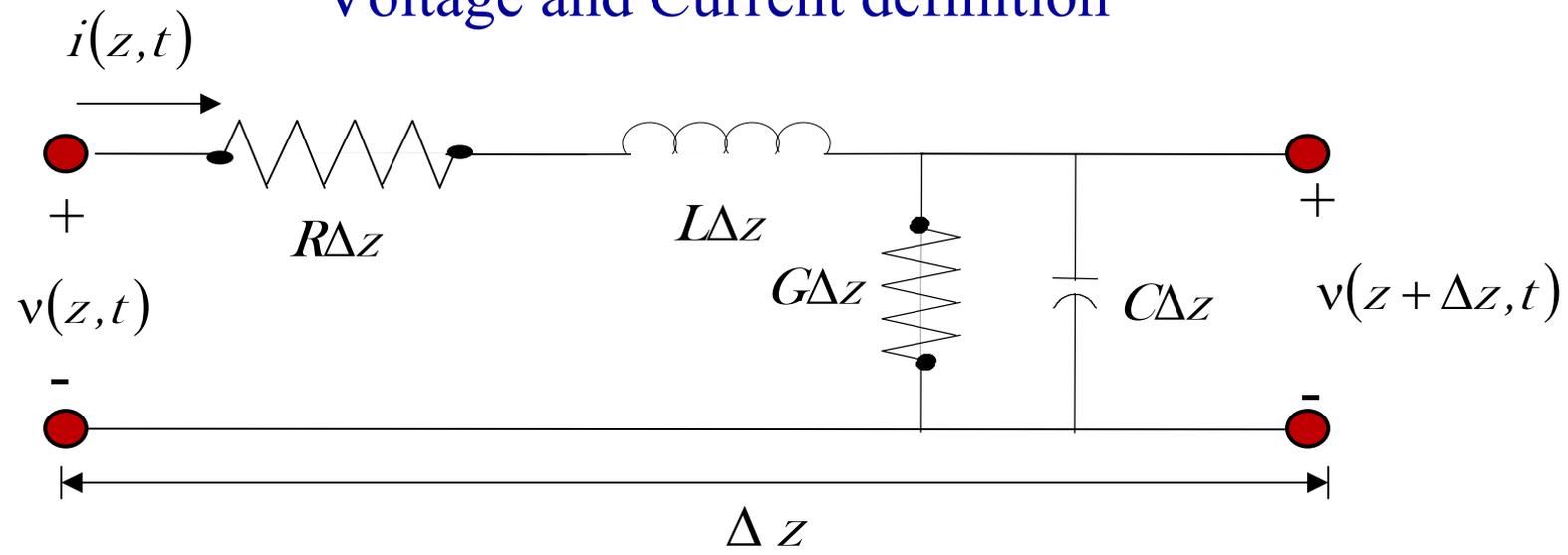
▣ Matched Impedance

◆ Complex Plane (Conformal Mapping) and Smith Chart

TL – Schematic Representation



Voltage and Current definition



Lumped-element equivalent circuit

Transmission Line Equations

Apply Kirchhoff's voltage and current laws:

$$v(z, t) = R\Delta z i(z, t) + L\Delta z \frac{\partial i(z, t)}{\partial t} + v(z + \Delta z, t)$$

$$i(z, t) = G\Delta z v(z + \Delta z, t) + C\Delta z \frac{\partial v(z + \Delta z, t)}{\partial t} + i(z + \Delta z, t)$$

Divide by Δz and taking the limit $\Delta z \rightarrow 0$:

$$\frac{\partial v(z, t)}{\partial z} = -Ri(z, t) - L \frac{\partial i(z, t)}{\partial t}$$

$$\frac{\partial i(z, t)}{\partial z} = -Gv(z, t) - C \frac{\partial v(z, t)}{\partial t}$$



$$\frac{\partial V(z)}{\partial z} = -(R + j\omega L)I(z)$$

$$\frac{\partial I(z)}{\partial z} = -(G + j\omega C)V(z)$$

Traveling Wave Solutions

$$\frac{\partial^2 V(z)}{\partial z^2} = -(R + j\omega L) \frac{\partial I(z)}{\partial z}$$

$$\frac{\partial I(z)}{\partial z} = -(G + j\omega C)V(z)$$

$$\frac{\partial^2 V(z)}{\partial z^2} = -(R + j\omega L)[-(G + j\omega C)V(z)]$$

$$\frac{\partial^2 V(z)}{\partial z^2} = (R + j\omega L)(G + j\omega C)V(z)$$

Traveling Wave Solutions

$$V(z) = V_0^+ e^{-\gamma z} + V_0^- e^{\gamma z}$$

$$\gamma = \alpha + j\beta = \sqrt{(R + j\omega L)(G + j\omega C)}$$

$$v(z, t) = |V_0^+| \cos(\omega t - \beta z + \phi^+) + |V_0^-| \cos(\omega t + \beta z + \phi^-)$$

$$I(z) = \frac{V_0^+}{Z_0} e^{-\gamma z} - \frac{V_0^-}{Z_0} e^{\gamma z}$$

$$\lambda = \frac{2\pi}{\beta}$$

$$v_p = \frac{\omega}{\beta} = \lambda f$$

$$Z_0 = \sqrt{\frac{R + j\omega L}{G + j\omega C}}$$

The Lossless Line

- In many practical cases, the loss of the line is very small and so can be ignored.

$$\gamma = j\beta = j\omega\sqrt{LC}$$

$$Z_0 = 1/Y_0 = \sqrt{\frac{L}{C}}$$

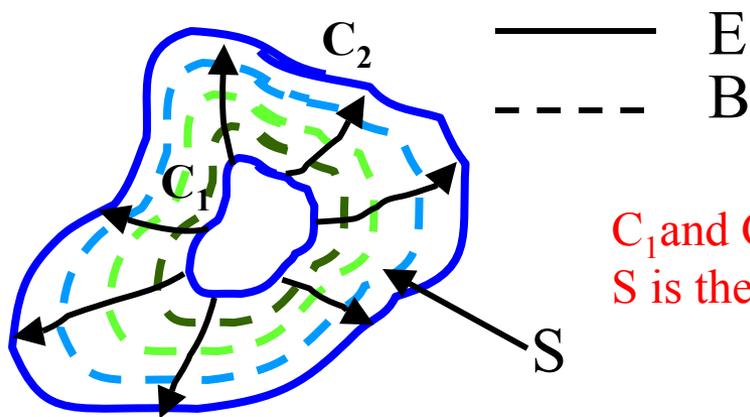
$$\lambda = \frac{2\pi}{\beta} = \frac{2\pi}{\omega\sqrt{LC}}$$

$$v_p = \frac{\omega}{\beta} = \frac{1}{\sqrt{LC}}$$

$$V(z) = V_0^+ e^{-\gamma z} + V_0^- e^{\gamma z}$$

$$I(z) = \frac{V_0^+}{Z_0} e^{-\gamma z} - \frac{V_0^-}{Z_0} e^{\gamma z}$$

Field Analysis of T.L.



C_1 and C_2 are line integration contours,
 S is the cross-sectional surface.

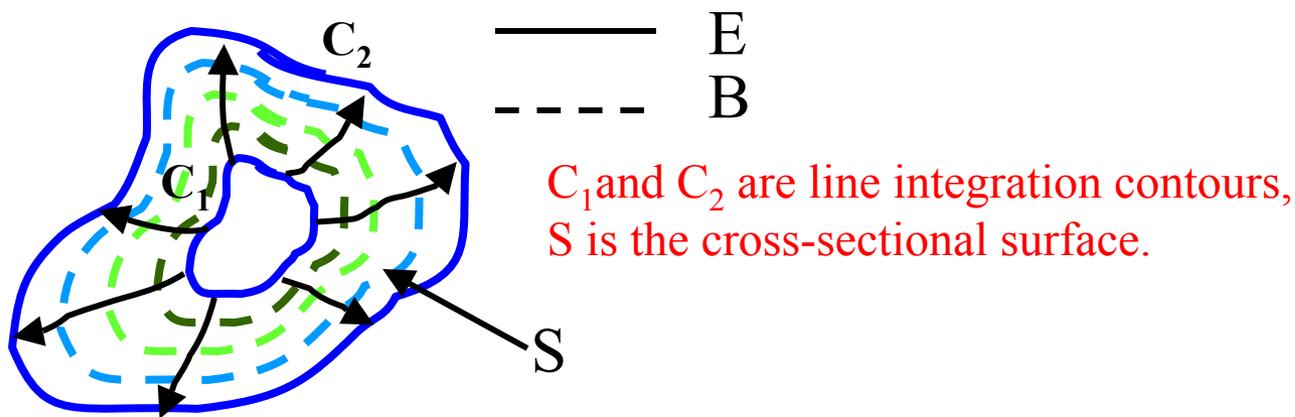
■ Field lines on an arbitrary TEM transmission line.

$$\text{Field Theory : } W_m = \frac{\mu_0}{4} \int_S \bar{H} \cdot \bar{H}^* ds \quad W_e = \frac{\epsilon_0}{4} \int_S \bar{E} \cdot \bar{E}^* ds$$

$$\text{Circuit Theory : } W_m = \frac{L|I_0|^2}{4} \quad W_e = \frac{C|V_0|^2}{4}$$

$$\text{Self Inductance : } L = \frac{\mu'}{|I_0|^2} \int_S \bar{H} \cdot \bar{H}^* ds \text{ H/m} \quad C = \frac{\epsilon'}{|V_0|^2} \int_S \bar{E} \cdot \bar{E}^* ds \text{ F/m}$$

Field Analysis of T.L.



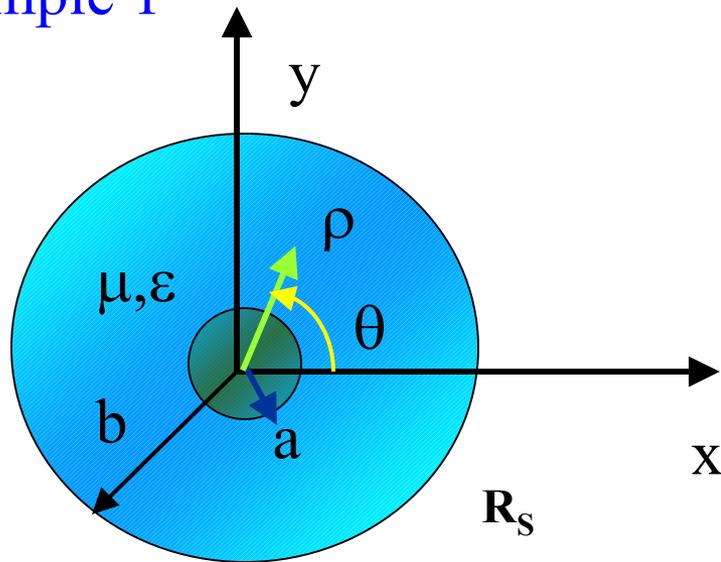
■ Field lines on an arbitrary TEM transmission line.

$$\text{Field Theory : } P_c = \frac{R_s}{2} \int_{C_1+C_2} \bar{H} \cdot \bar{H}^* dl \quad P_d = \frac{\epsilon''}{2} \int_S \bar{E} \cdot \bar{E}^* ds$$

$$\text{Circuit Theory : } P_c = \frac{R |I_0|^2}{2} \quad P_d = \frac{G |V_0|^2}{2}$$

$$R/G \text{ per Meter } R = \frac{R_s}{|I_0|^2} \int_{C_1+C_2} \bar{H} \cdot \bar{H}^* dl \quad \Omega/m \quad G = \frac{\omega \epsilon''}{|V_0|^2} \int_S \bar{E} \cdot \bar{E}^* ds \quad S/m$$

Example 1



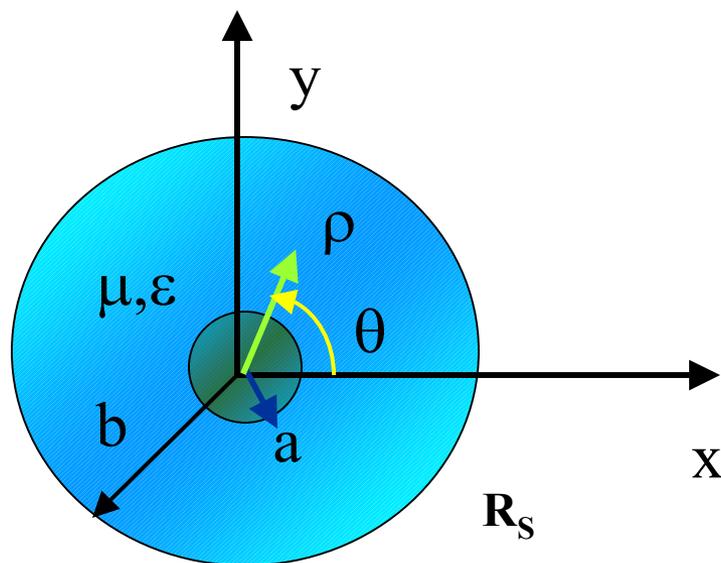
$$\bar{E} = \frac{V_0}{\rho \ln(b/a)} e^{-\gamma z} \hat{\rho}$$

$$\bar{H} = \frac{I_0}{2\pi\rho} e^{-\gamma z} \hat{\phi}$$

■ Geometry of a coaxial line with surface resistance R_s on the inner and outer conductors.

- γ is the propagation constant of the line.
- R_s is the surface resistivity.
- $\epsilon = \epsilon' - j\epsilon''$
- $\mu = \mu_0\mu_r$

Example 1 – Calculation of L

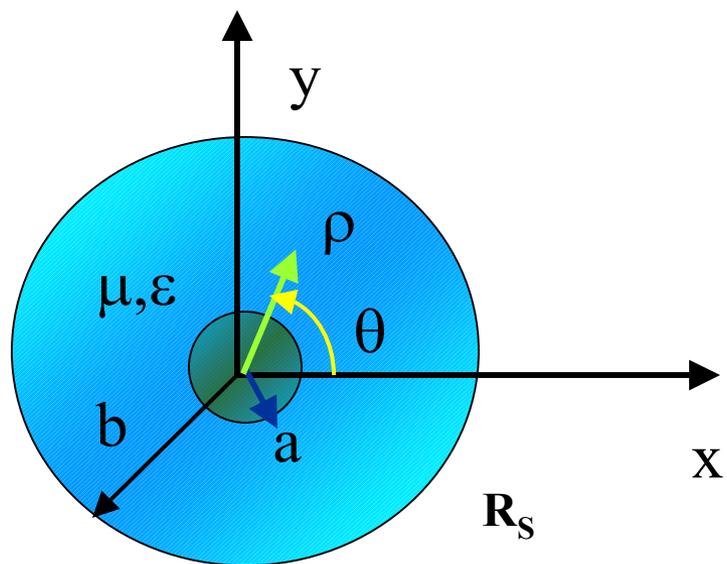


$$\bar{H} = \frac{I_0}{2\pi\rho} e^{-\gamma z} \hat{\phi}$$

$$L = \frac{\mu}{|I_0|^2} \int_S \bar{H} \cdot \bar{H}^* ds = \frac{\mu}{(2\pi)^2} \int_0^{2\pi} \int_a^b \frac{1}{\rho^2} \rho d\rho d\phi$$

$$= \frac{\mu}{2\pi} \ln\left(\frac{b}{a}\right)$$

Example 1 – Calculation of R



$$\bar{H} = \frac{I_0}{2\pi\rho} e^{-\gamma z} \hat{\phi}$$

$$R = \frac{R_S}{|I_0|^2 c_1 + c_2} \int \bar{H} \cdot \bar{H}^* d\ell = \frac{R_S}{(2\pi)^2} \left(\int_{\phi=0}^{2\pi} \frac{1}{a^2} a d\phi + \int_{\phi=0}^{2\pi} \frac{1}{b^2} b d\phi \right)$$

$$= \frac{R_S}{2\pi} \left(\frac{1}{a} + \frac{1}{b} \right)$$

Lossless Coaxial Line

⊕ Propagation Constant



$$\beta = \omega\sqrt{\mu\epsilon} = \omega\sqrt{LC}$$

⊕ Wave Impedance



$$Z_w = \frac{E_\rho}{H_\phi} = \frac{\omega\mu}{\beta} = \sqrt{\frac{\mu}{\epsilon}} = \eta$$

⊕ Characteristic Impedance



$$Z_0 = \frac{V_0}{I_0} = \frac{E_\rho \ln(b/a)}{2\pi H_\phi}$$

⊕ Power Flow



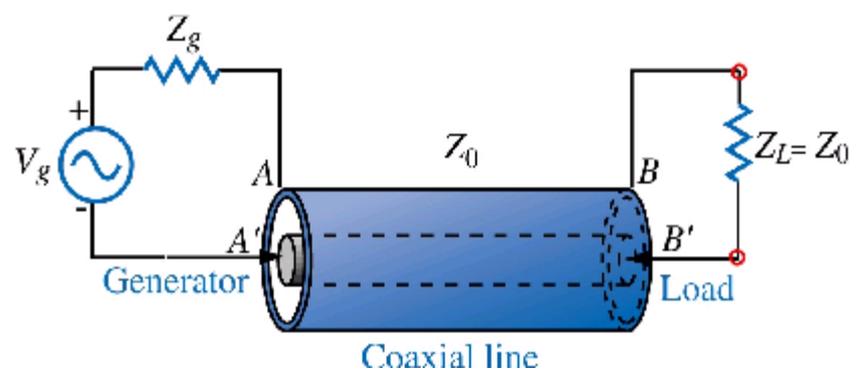
$$P = \frac{1}{2} \int_s \bar{E} \times \bar{H}^* ds = \frac{1}{2} V_0 I_0^*$$

◆ The flow of power in a transmission line takes place entirely via the electric and magnetic fields between the two conductors; *power is not transmitted through the conductors themselves.*

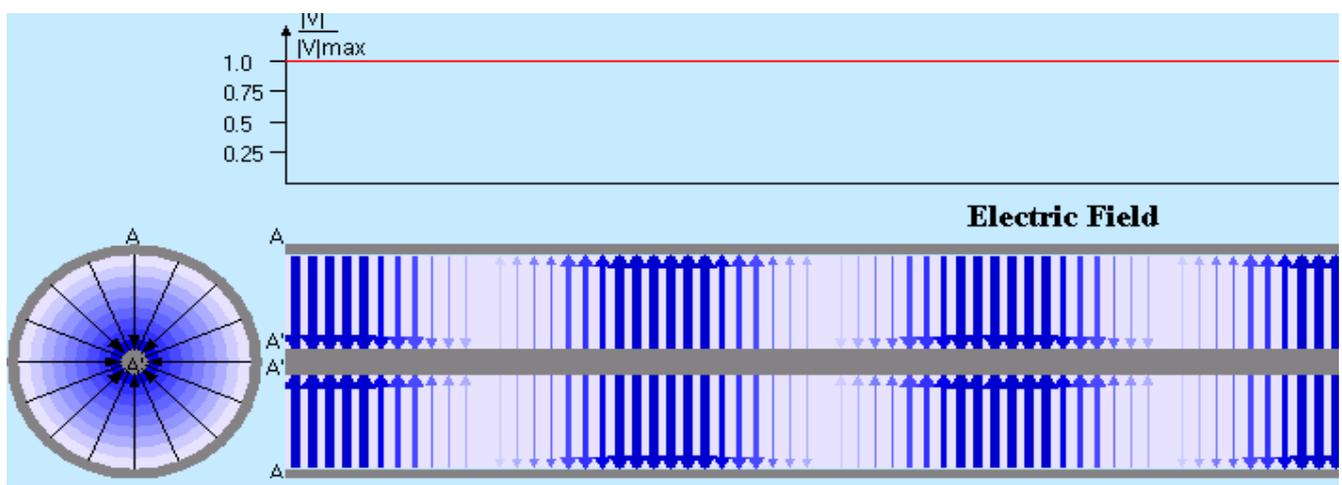
◆ For the case of finite conductivity, power may enter the conductors, but this power is then lost as heat and is not delivered to the load.

Coaxial cable as a transmission line with TEM mode

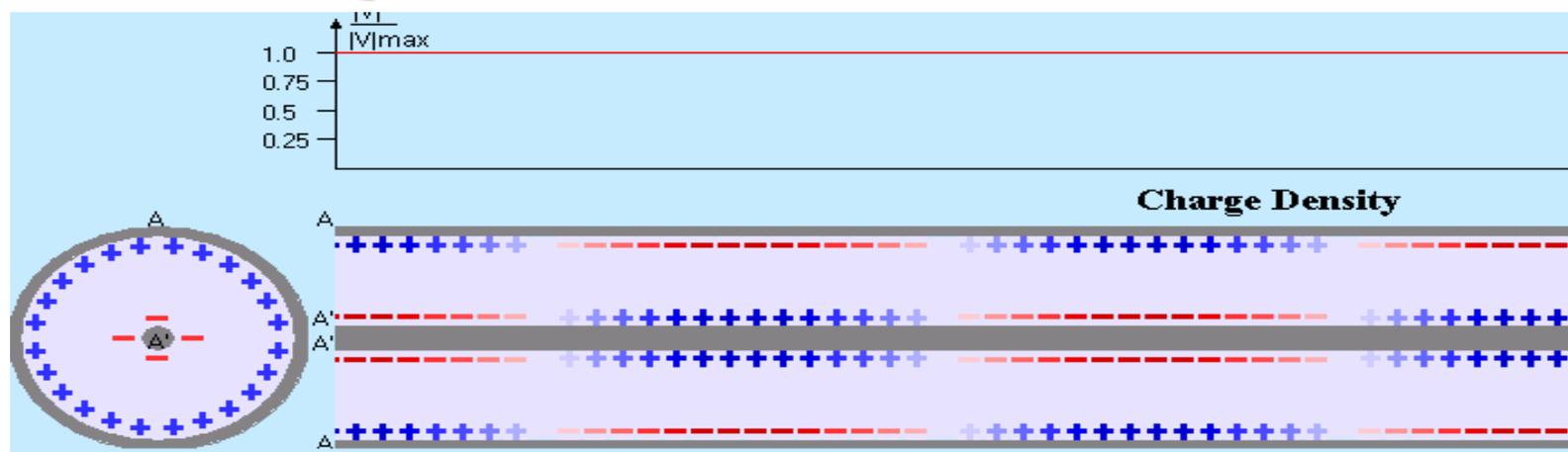
When coaxial cable is terminated by characteristic impedance, the line is perfectly matched and the voltage is constant along the line. The VSWR=1.



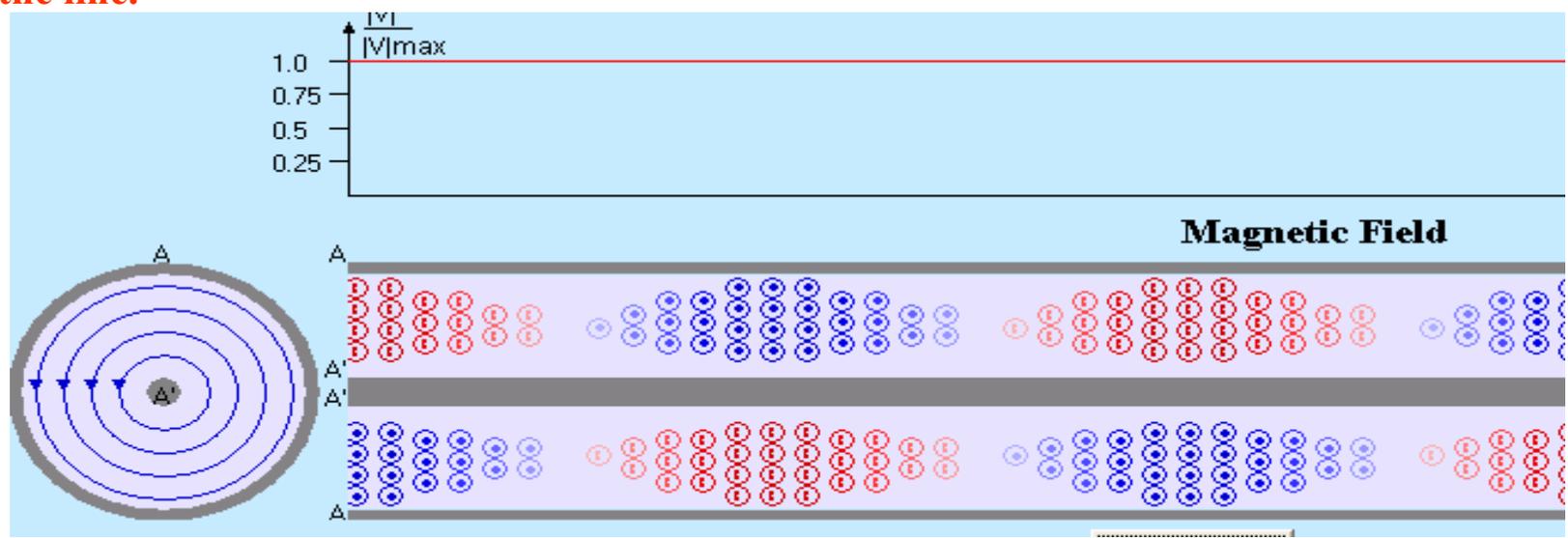
Coaxial line terminated by characteristic impedance.



E-field in coaxial line. Orientation depends on phase - position along the line.



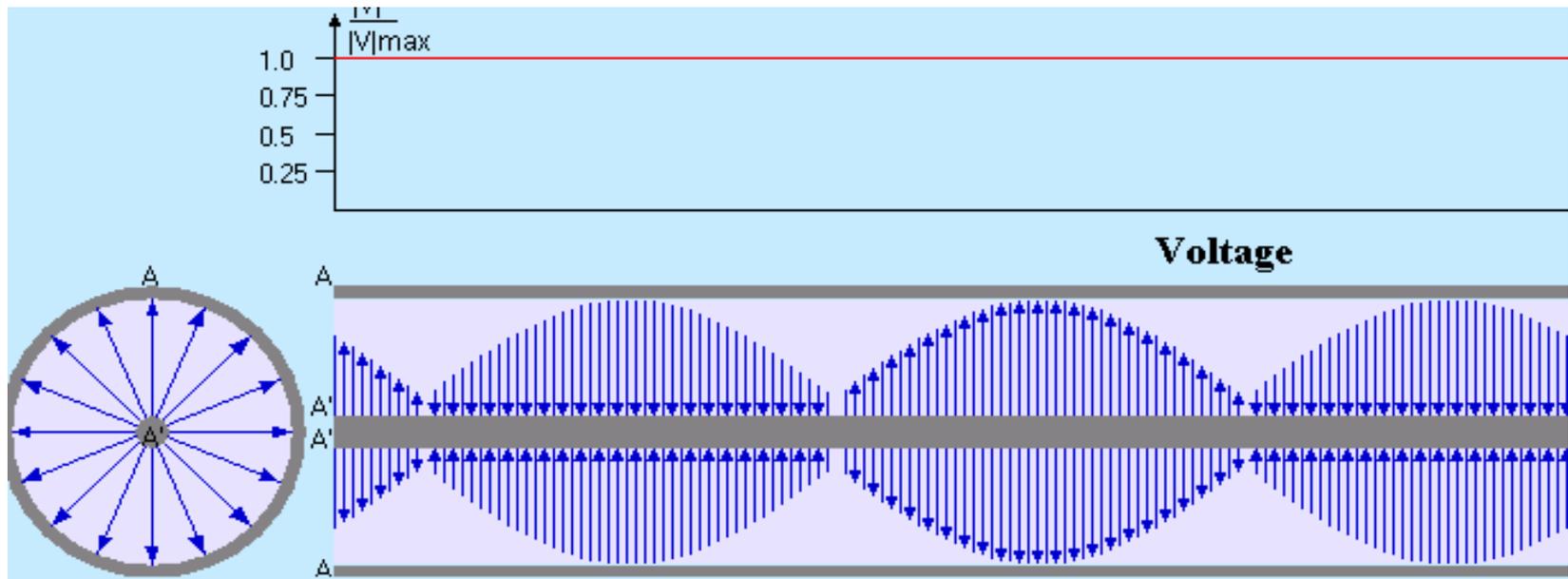
Surface charge density induced in coaxial line. The sign depends on phase - position along the line.



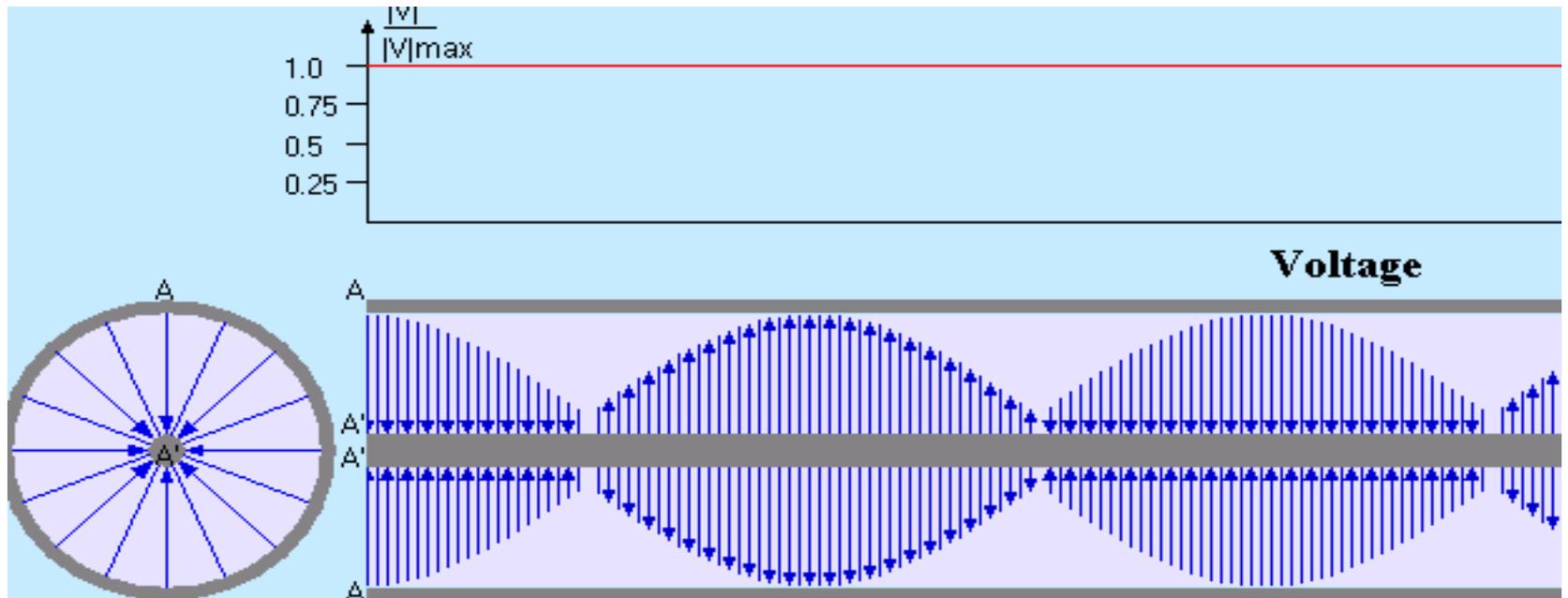
H-field in coaxial line. Orientation depends on phase - position along the line



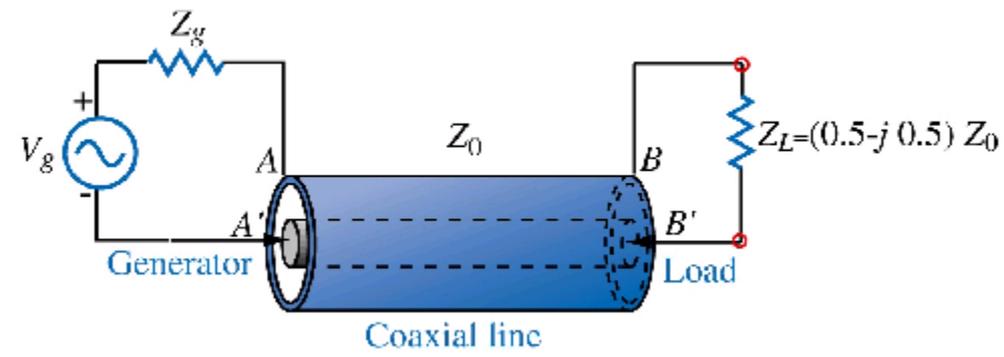
Surface current in coaxial line. Orientation depends on phase - position along the line.



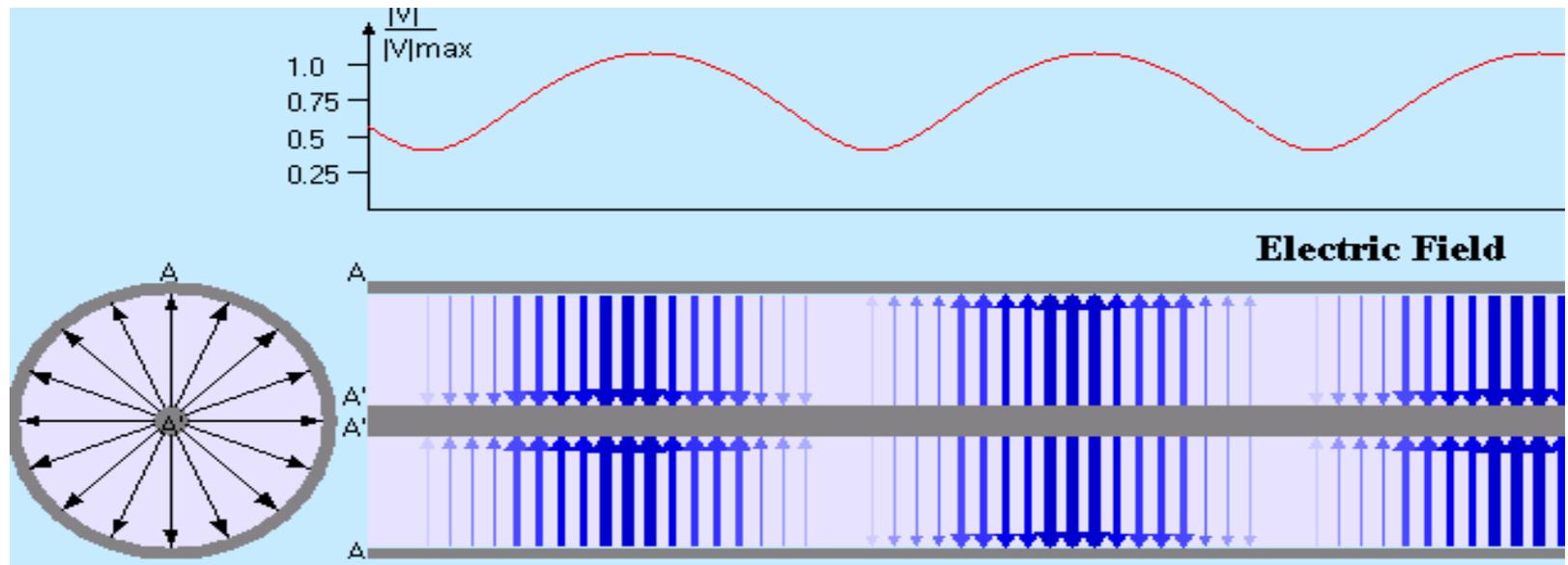
Voltage on coaxial line does not depend on time and position, because the load is matched.



Voltage on coaxial line does not depend on time and position, because the load is matched.

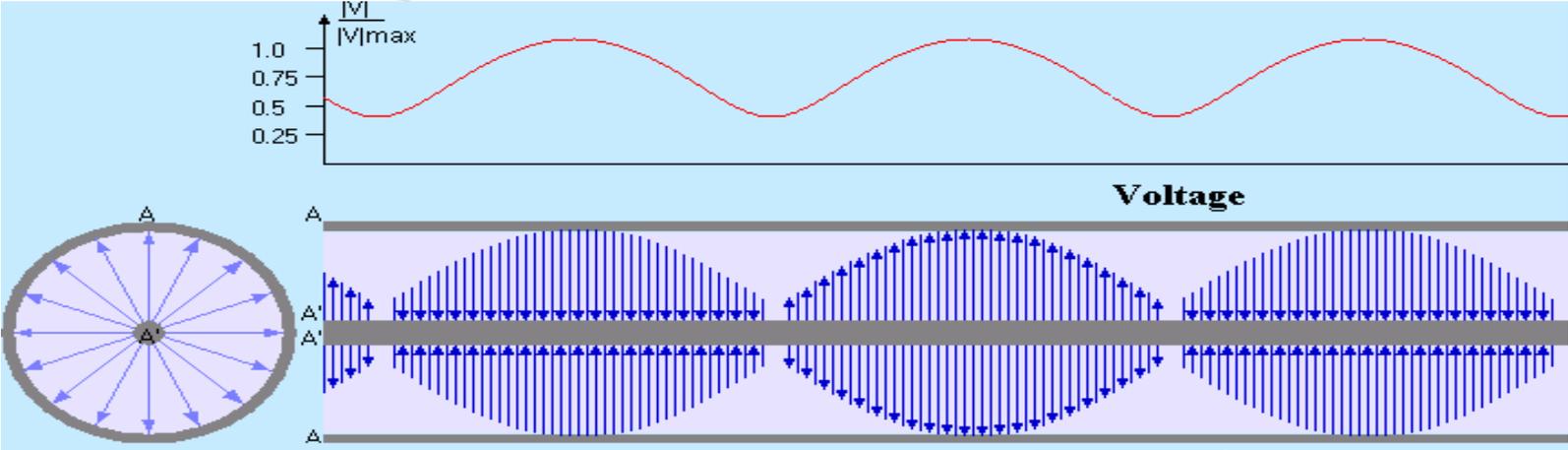


Coaxial line terminated by an impedance different than the characteristic impedance.

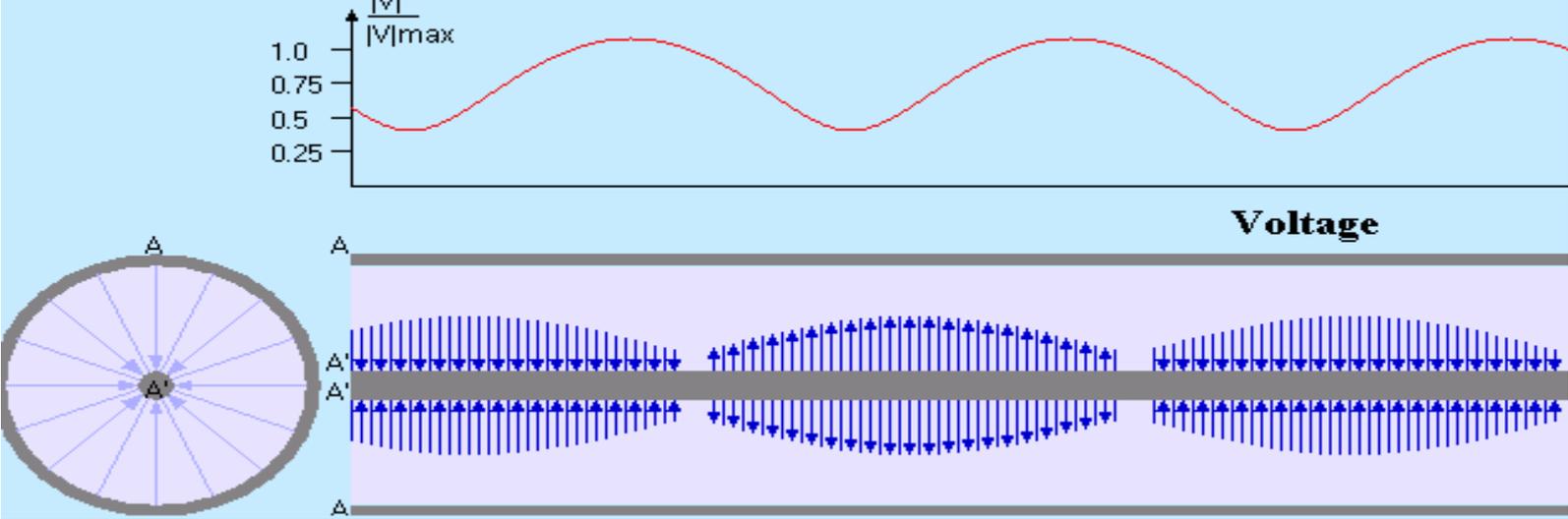


E-field in coaxial line. Orientation depends on phase - position along the line. The VSWR depends on reflection coefficient.

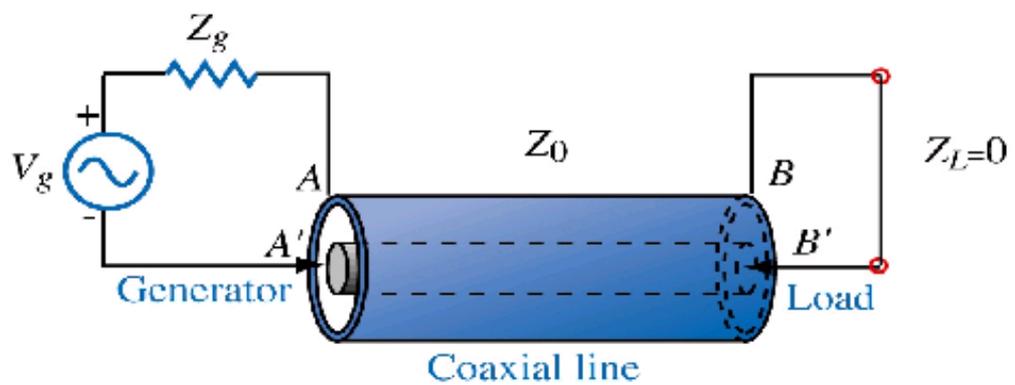
advanced photon source



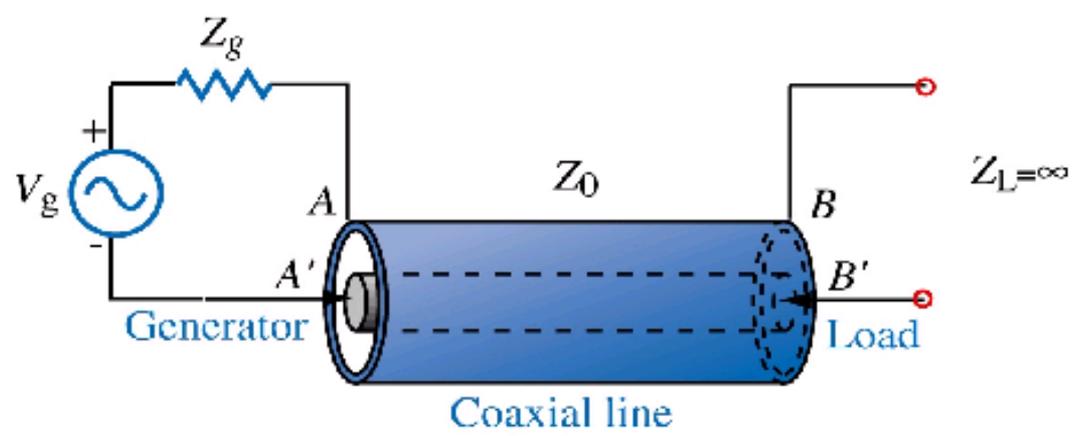
Voltage on coaxial line depends on time and position.



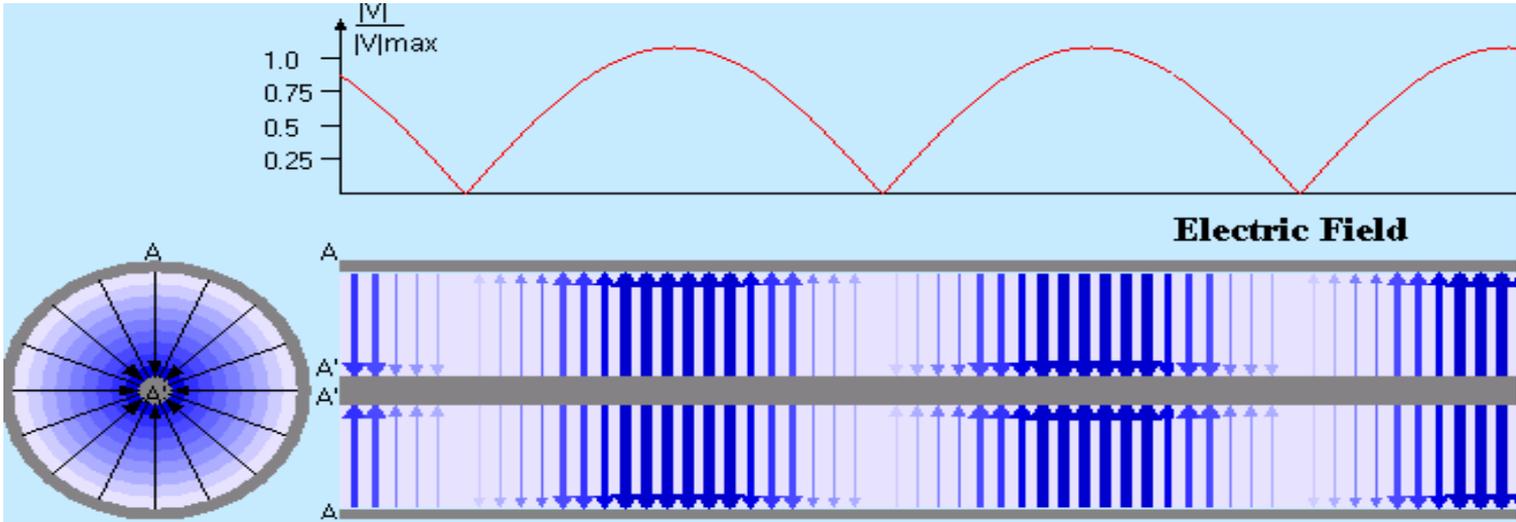
Voltage on coaxial line depends on time and position, but does not go to zero.



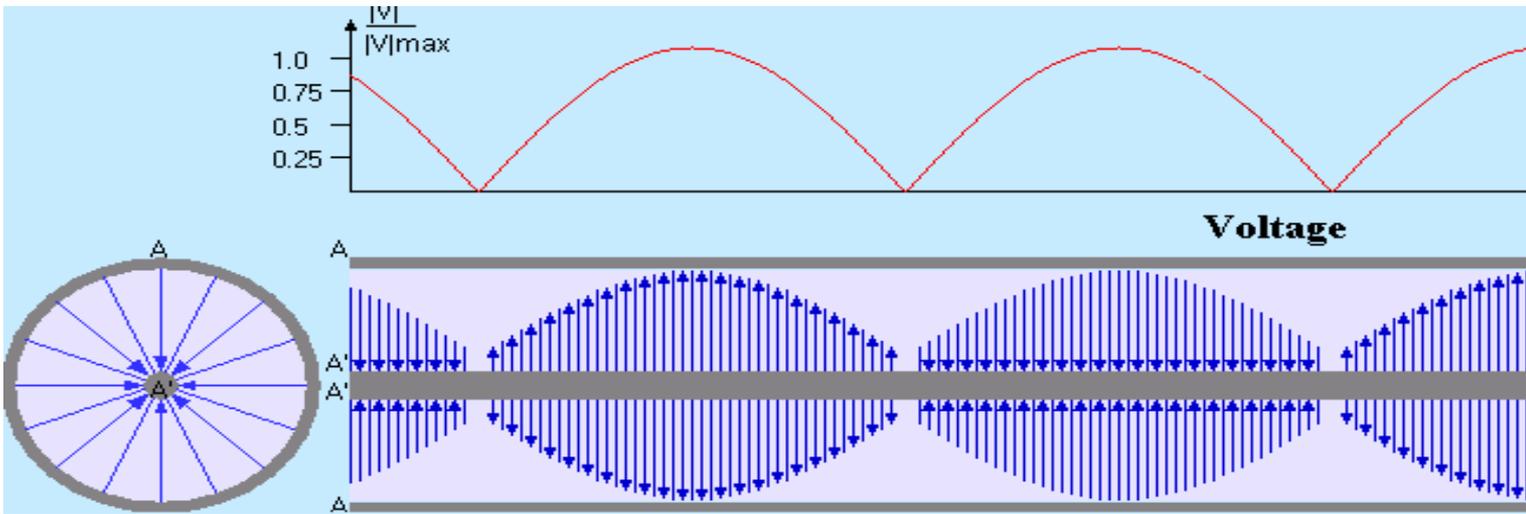
Coaxial line terminated by short circuit. The VSWR=indefinitely large.



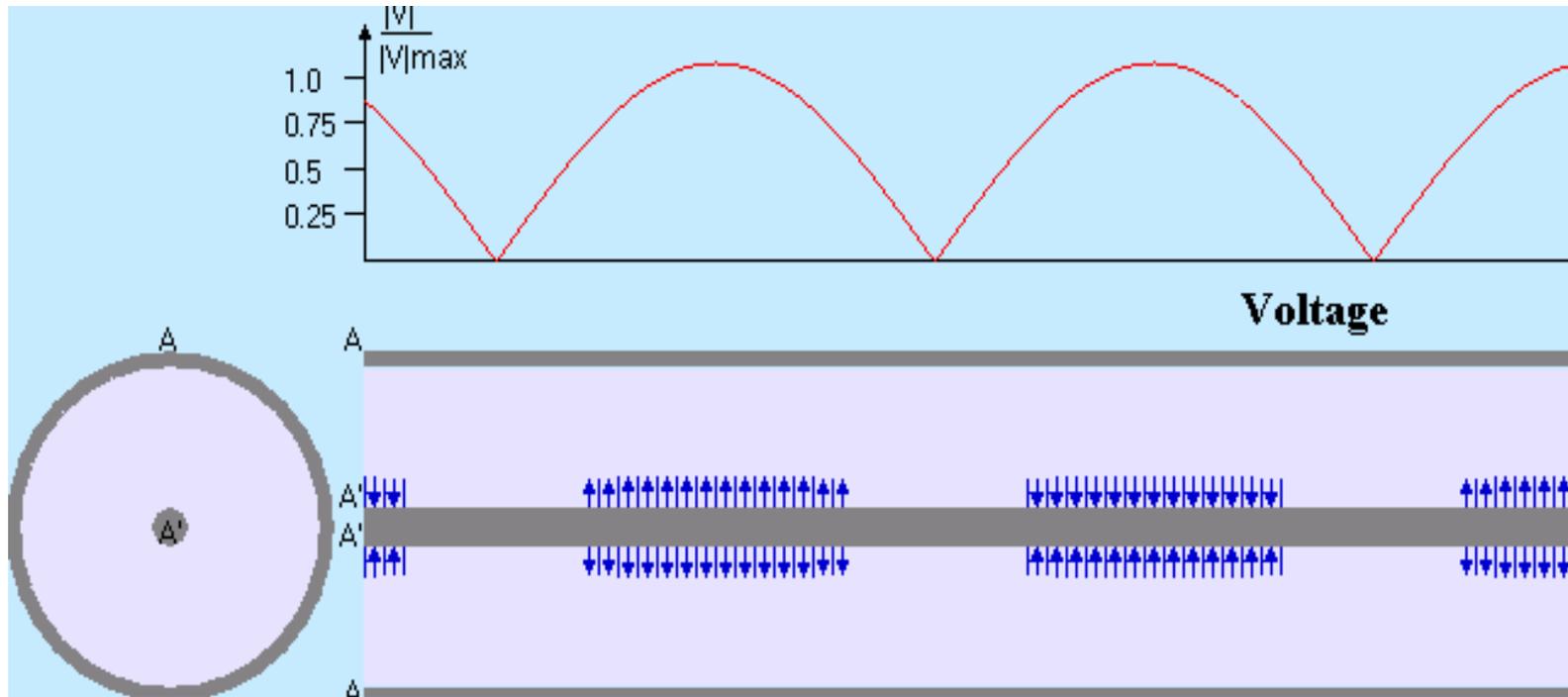
Coaxial line terminated by open circuit. The VSWR=indefinitely large.



E-field in coaxial line. Orientation depends on phase - position along the line.



Voltage on coaxial line depends on time and position. The VSWR=indefinitely large.



Voltage on coaxial line depends on time and position, and at nodes, goes to zero .

Special Cases

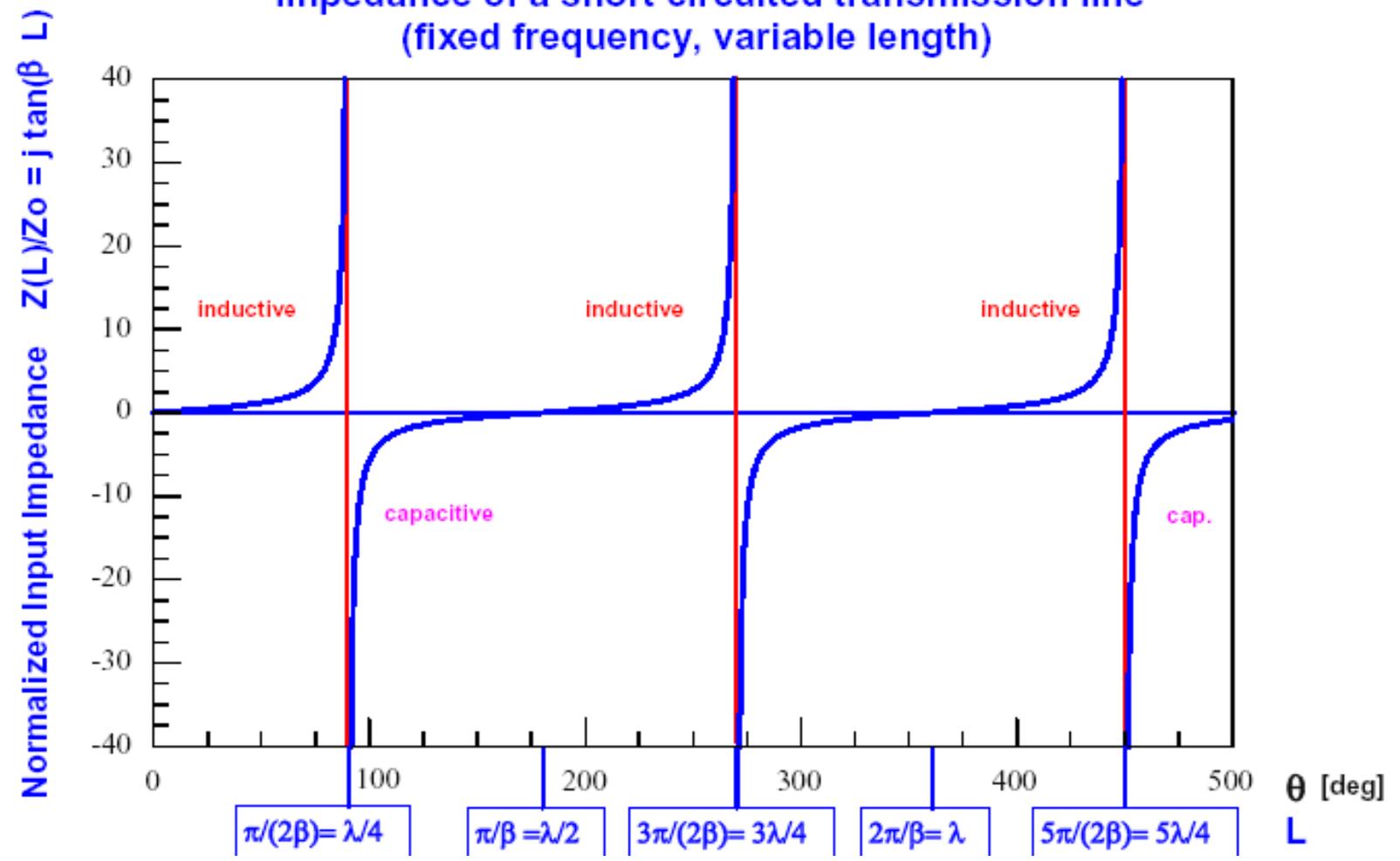
$Z_R \rightarrow 0$ (SHORT CIRCUIT)



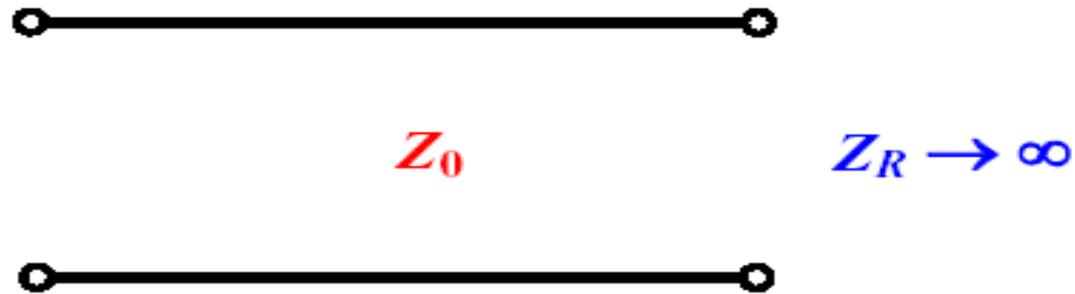
The load **boundary condition** due to the short circuit is $V(0) = 0$

$$\begin{aligned} \Rightarrow V(d = 0) &= V^+ e^{j\beta 0} (1 + \Gamma_R e^{-j2\beta 0}) \\ &= V^+ (1 + \Gamma_R) = 0 \\ \Rightarrow \Gamma_R &= -1 \end{aligned}$$

Impedance of a short circuited transmission line
(fixed frequency, variable length)



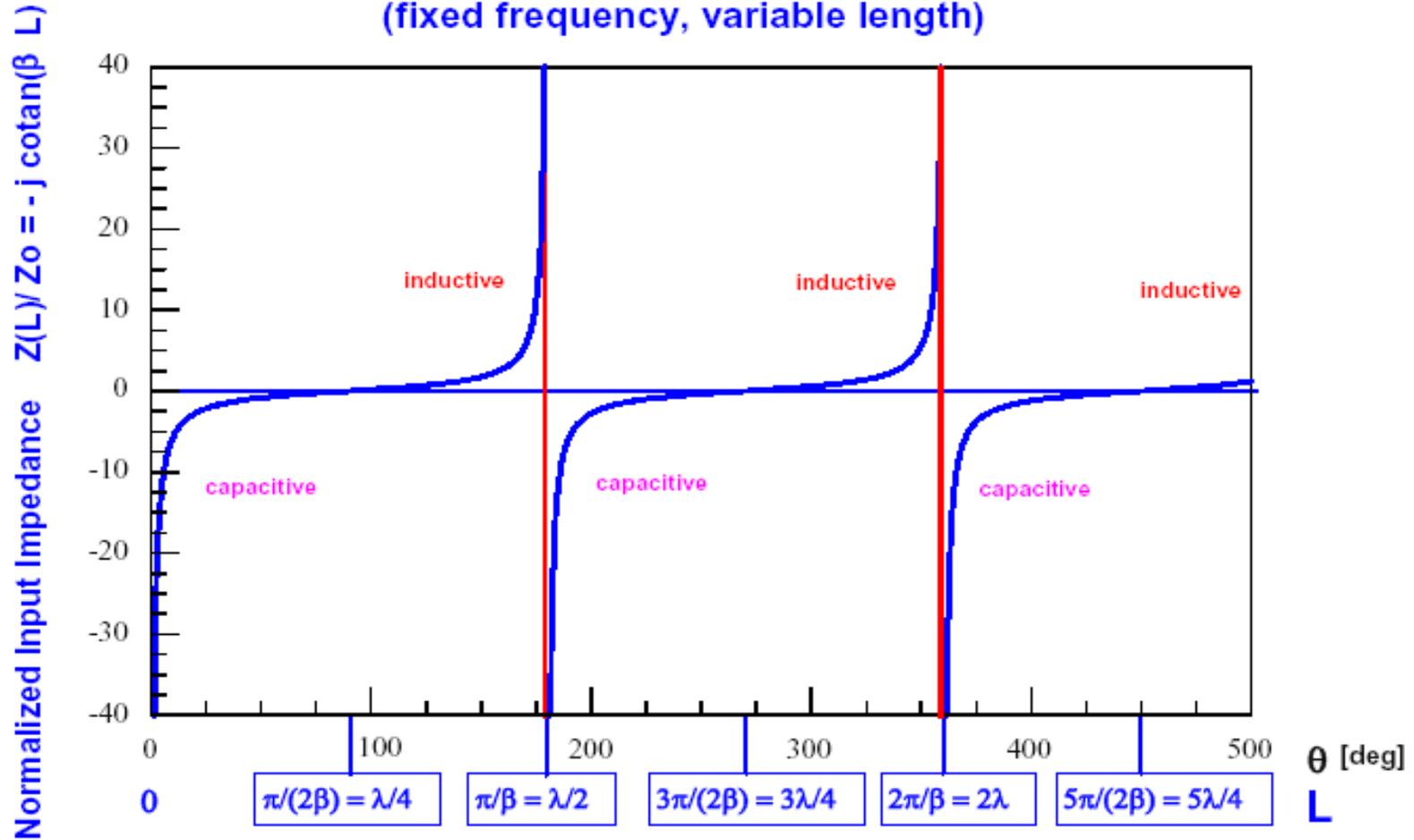
$Z_R \rightarrow \infty$ (OPEN CIRCUIT)



The load **boundary condition** due to the open circuit is $I(0) = 0$

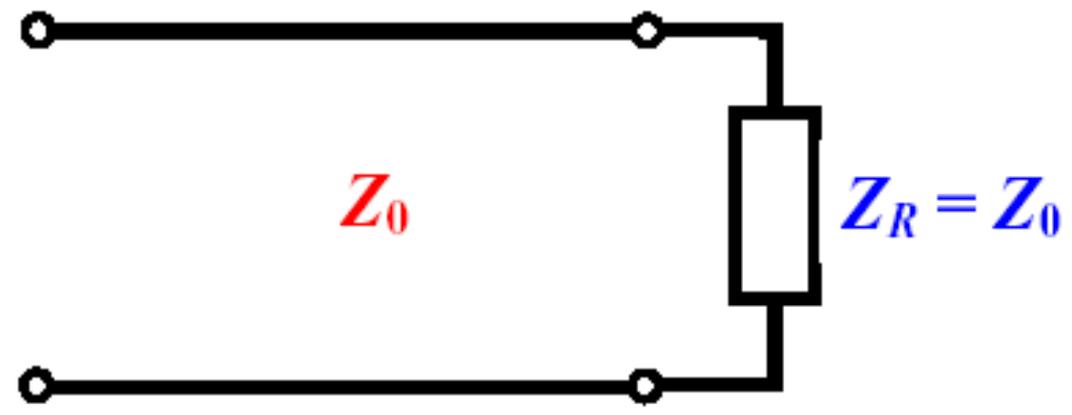
$$\begin{aligned} \Rightarrow I(d = 0) &= \frac{V^+}{Z_0} e^{j\beta 0} (1 - \Gamma_R e^{-j2\beta 0}) \\ &= \frac{V^+}{Z_0} (1 - \Gamma_R) = 0 \\ \Rightarrow \Gamma_R &= 1 \end{aligned}$$

Impedance of an open circuited transmission line
(fixed frequency, variable length)



Line length L

$$Z_R = Z_0 \text{ (MATCHED LOAD)}$$



The reflection coefficient for a matched load is

$$\Gamma_R = \frac{Z_R - Z_0}{Z_R + Z_0} = \frac{Z_0 - Z_0}{Z_0 + Z_0} = 0$$

no reflection!

Standing Wave Patterns

In practical applications it is very convenient to plot the **magnitude** of **phasor voltage** and **phasor current** along the transmission line. These are the **standing wave patterns**:

$$\text{Loss - less line} \left\{ \begin{array}{l} |V(d)| = |V^+| \cdot |(1 + \Gamma(d))| \\ |I(d)| = \left| \frac{V^+}{Z_0} \right| \cdot |(1 - \Gamma(d))| \end{array} \right.$$

We limit now our discussion to the **loss-less** transmission line case where the **generalized reflection coefficient** varies as

$$\Gamma(d) = \Gamma_R \exp(-j2\beta d) = |\Gamma_R| \exp(j\phi) \exp(-j2\beta d)$$

Note that the magnitude of an **exponential** with **imaginary argument** is always **unity**, so

$$|\exp(j\phi) \exp(-j2\beta d)| = 1$$

and that in a loss-less line it is **always** true that, at any line location,

$$|\Gamma(d)| = |\Gamma_R|$$

When **d** increases, moving from load to generator, the **generalized reflection coefficient** on the complex plane moves **clockwise** on a circle with radius **$|\Gamma_R|$** and is identified by the angle **$\phi - 2\beta d$** .

The **voltage standing wave pattern** has a **maximum** at locations where the generalized reflection coefficient is **real** and **positive**

$$\Gamma(d) = |\Gamma_R| \exp(j\phi) \exp(-j2\beta d) = 1 \quad \Rightarrow \quad |\phi - 2\beta d| = 2n\pi$$

At these locations we have

$$|1 + \Gamma(d)| = 1 + |\Gamma_R|$$

$$\Rightarrow V_{\max} = |V(d_{\max})| = |V^+| \cdot (1 + |\Gamma_R|)$$

The phase angle $\phi - 2\beta d$ changes by an amount 2π , when moving from one maximum to the next. This corresponds to a distance between successive maxima of $\lambda/2$.

The **voltage standing wave pattern** has a **minimum** at locations where the generalized reflection coefficient is **real** and **negative**

$$\Gamma(d) = -|\Gamma_R|$$

$$\exp(j\phi)\exp(-j2\beta d) = -1 \quad \Rightarrow \quad |\phi - 2\beta d| = (2n + 1)\pi$$

At these locations we have

$$|1 + \Gamma(d)| = 1 - |\Gamma_R|$$

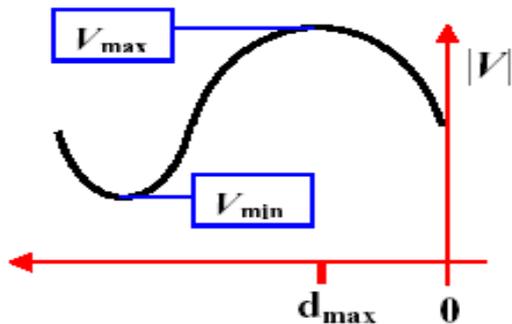
$$\Rightarrow V_{\min} = |V(d_{\min})| = |V^+| \cdot (1 - |\Gamma_R|)$$

Also when moving from one minimum to the next, the phase angle $\phi - 2\beta d$ changes by an amount 2π . This again corresponds to a distance between successive minima of $\lambda/2$.

- ❑ STEP 1: The VSWR provides the magnitude of the load reflection coefficient

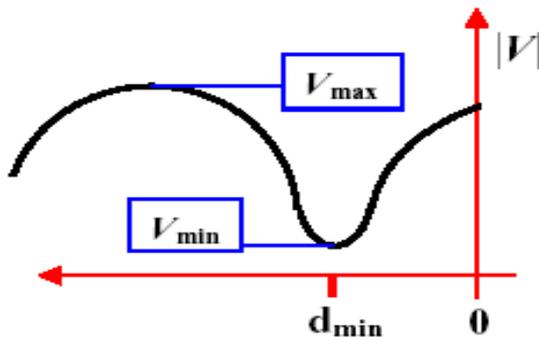
$$|\Gamma_R| = \frac{VSWR - 1}{VSWR + 1}$$

- ❑ STEP 2: The distance from the load of the first maximum or minimum gives the phase ϕ of the load reflection coefficient.



For an inductive reactance, a voltage maximum is closest to the load and

$$\phi = 2\beta d_{\max} = \frac{4\pi}{\lambda} d_{\max}$$



For a capacitive reactance, a voltage minimum is closest to the load and

$$\phi = -\pi + 2\beta d_{\min} = -\pi + \frac{4\pi}{\lambda} d_{\min}$$

- **STEP 3:** The load impedance is obtained by inverting the expression for the reflection coefficient

$$\Gamma_R = |\Gamma_R| \exp(j\phi) = \frac{Z_R - Z_0}{Z_R + Z_0}$$

$$\Rightarrow Z_R = Z_0 \frac{1 + |\Gamma_R| \exp(j\phi)}{1 - |\Gamma_R| \exp(j\phi)}$$

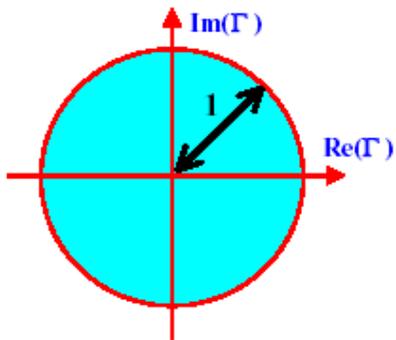
The voltage standing wave pattern provides immediate information on the transmission line circuit

- ❑ If the load is matched to the transmission line ($Z_R = Z_0$) the voltage standing wave pattern is flat, with value $|V^+|$.
- ❑ If the load is real and $Z_R > Z_0$, the voltage standing wave pattern starts with a maximum at the load.
- ❑ If the load is real and $Z_R < Z_0$, the voltage standing wave pattern starts with a minimum at the load.
- ❑ If the load is complex and $\text{Im}(Z_R) > 0$ (inductive reactance), the voltage standing wave pattern initially increases when moving from load to generator and reaches a maximum first.
- ❑ If the load is complex and $\text{Im}(Z_R) < 0$ (capacitive reactance), the voltage standing wave pattern initially decreases when moving from load to generator and reaches a minimum first.

Smith Chart

- ◆ Smith Chart was developed in 1939 by P. Smith at the Bell Telephone Laboratory.
- ◆ One can develop intuition about transmission line and impedance-matching problems by learning to think in terms of the Smith Chart.
- ◆ Smith Chart is essentially a plot of the voltage reflection coefficient, Γ , in the complex plane.
- ◆ It can be used to convert from voltage reflection coefficient (Γ) to normalized impedances ($z=Z/Z_0$) and admittances ($y=Y/Y_0$), and vice versa.

Smith Charts



The domain of definition of the reflection coefficient is a circle of radius 1 in the complex plane. This is also the domain of the Smith chart.

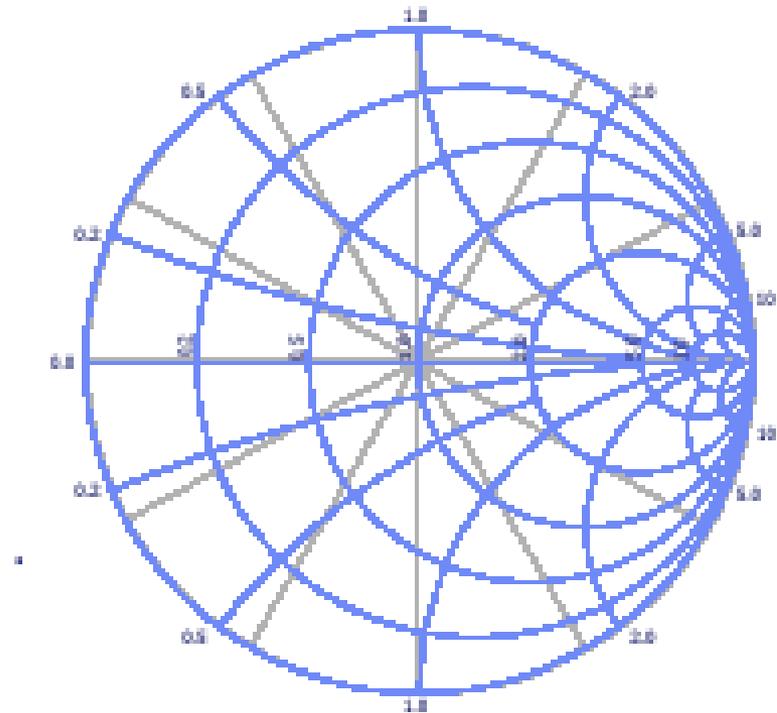
Complex Γ Plane with z Circles

- ◆ The Smith Chart is a plot of the voltage reflection coefficient, Γ , on the complex plane superimposed with impedance circles.
- ◆ The complex Γ plane.
- ◆ The impedance circles.

$$z = \frac{Z}{Z_0} = r + jx$$

$$y = \frac{Y}{Y_0} = g + jb$$

$$z = \frac{1}{y}$$



Conformal Mapping - Γ and Z

- If a lossless line of characteristic impedance Z_0 is terminated with a load impedance Z_L , the reflection coefficient at the load can be written as:

$$\Gamma = \frac{Z_L - Z_0}{Z_L + Z_0} = |\Gamma| e^{j\theta} \quad \text{where } Z_L = \frac{Z_L}{Z_0}$$

- This relation can be solved for Z_L in terms of Γ to give:

$$Z_L = \frac{1 + |\Gamma| e^{j\theta}}{1 - |\Gamma| e^{j\theta}} = \frac{(1 + \Gamma_r) + j\Gamma_i}{(1 - \Gamma_r) - j\Gamma_i} = r_L + jx_L$$

- The real and imaginary part of the above equation can be found by multiplying the numerator and denominator by the complex conjugate of the denominator to give:

$$r_L = \frac{1 - \Gamma_r^2 - \Gamma_i^2}{(1 - \Gamma_r)^2 + \Gamma_i^2}$$

$$x_L = \frac{2\Gamma_i}{(1 - \Gamma_r)^2 + \Gamma_i^2}$$

The r_L Circles

$$r_L = \frac{1 - \Gamma_r^2 - \Gamma_i^2}{(1 - \Gamma_r)^2 + \Gamma_i^2}$$

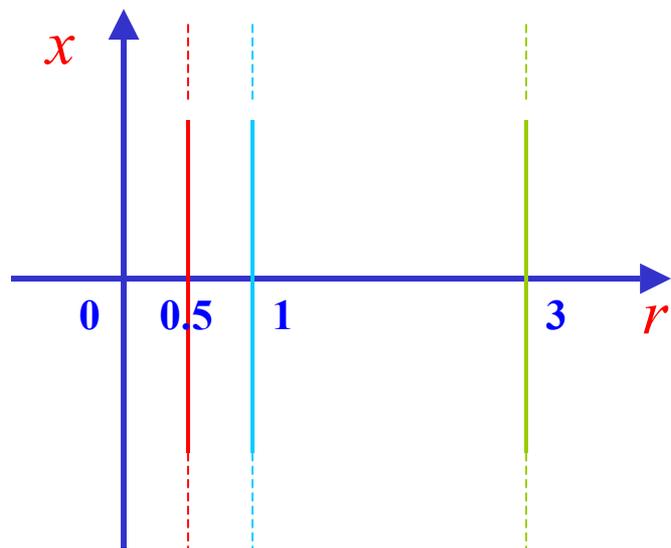
$$r_L - 2r_L\Gamma_r + r_L\Gamma_r^2 + r_L\Gamma_i^2 = 1 - \Gamma_r^2 - \Gamma_i^2$$

$$\frac{r_L}{1 + \Gamma_r} - \frac{2r_L\Gamma_r}{1 + \Gamma_r} + \Gamma_r^2 + \Gamma_i^2 = 1$$

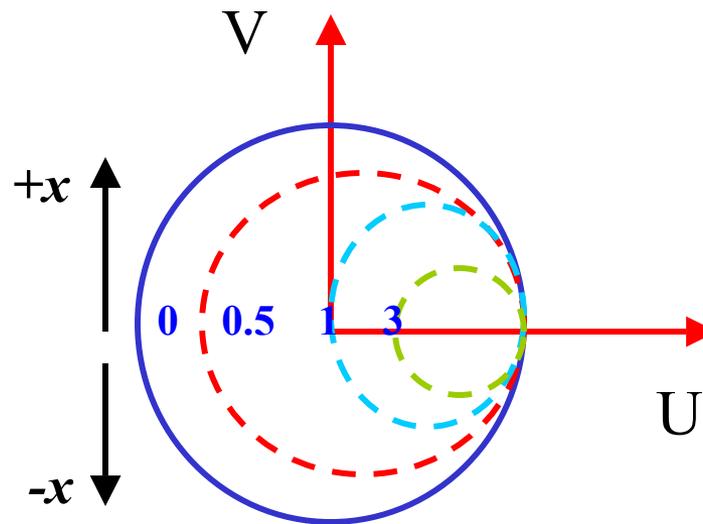
$$\frac{r_L}{1 + \Gamma_r} - \left(\frac{r_L}{1 + \Gamma_r}\right)^2 + \left(\frac{r_L}{1 + \Gamma_r}\right)^2 - \frac{2r_L\Gamma_r}{1 + \Gamma_r} + \Gamma_r^2 + \Gamma_i^2 = 1$$

$$\left(\Gamma_r - \frac{r_L}{1 + \Gamma_r}\right)^2 + \Gamma_i^2 = \left(\frac{1}{1 + \Gamma_r}\right)^2$$

The r_L Circles



Constant resistance lines in the $z = r + jx$



Γ plane

The x_L Circles

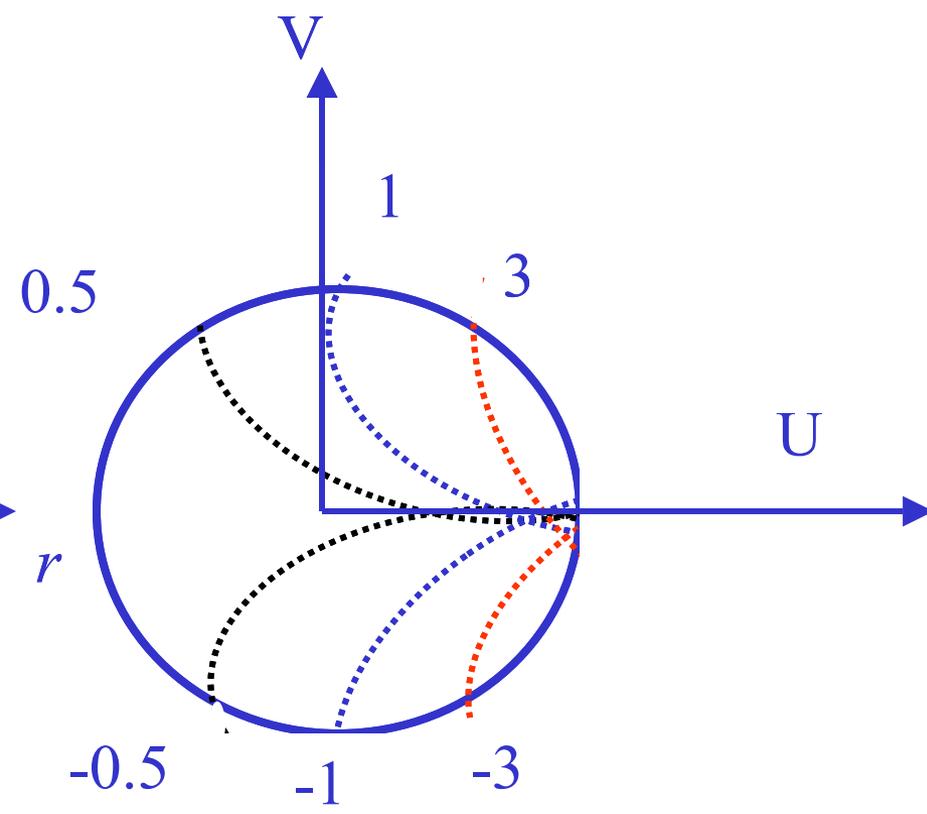
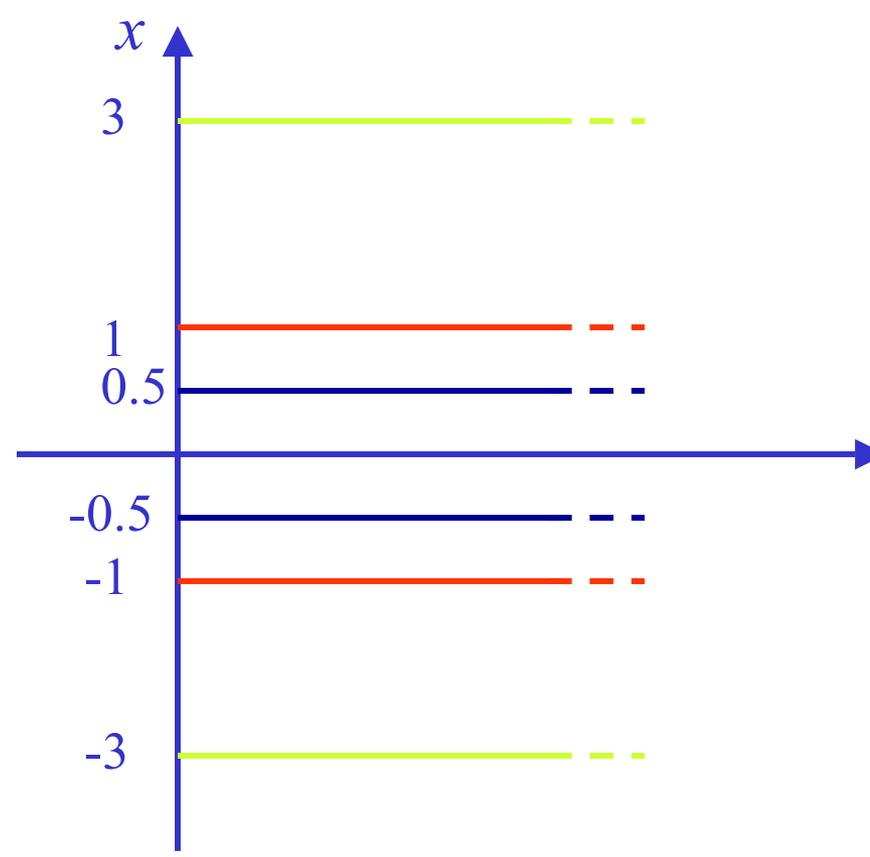
$$x_L = \frac{2\Gamma_i}{(1-\Gamma_r)^2 + \Gamma_i^2}$$

$$(1-\Gamma_r)^2 + \Gamma_i^2 = \frac{2\Gamma_i}{x_L}$$

$$(1-\Gamma_r)^2 + \Gamma_i^2 - \frac{2}{x_L} + \left(\frac{1}{x_L}\right)^2 = \left(\frac{1}{x_L}\right)^2$$

$$(1-\Gamma_r)^2 + \left(\Gamma_i - \frac{1}{x_L}\right)^2 = \left(\frac{1}{x_L}\right)^2$$

The x_L Circles



Constant reactance lines in (for $r \geq 0$)
in the $z = r + jx$

Γ plane

The goal of the Smith chart is to identify **all** possible **impedances** on the domain of existence of the reflection coefficient. To do so, we start from the general definition of **line impedance** (which is equally applicable to the load impedance)

$$Z(d) = \frac{V(d)}{I(d)} = Z_0 \frac{1 + \Gamma(d)}{1 - \Gamma(d)}$$

This provides the complex function $Z(d) = f\{\text{Re}(\Gamma), \text{Im}(\Gamma)\}$ that we want to graph. It is obvious that the result would be applicable only to lines with exactly characteristic impedance Z_0 .

In order to obtain **universal** curves, we introduce the concept of **normalized impedance**

$$z(d) = \frac{Z(d)}{Z_0} = \frac{1 + \Gamma(d)}{1 - \Gamma(d)}$$

The **normalized impedance** is represented on the **Smith chart** by using families of curves that identify the **normalized resistance** r (real part) and the **normalized reactance** x (imaginary part)

$$z(d) = \text{Re}(z) + j\text{Im}(z) = r + jx$$

Let's represent the **reflection coefficient** in terms of its coordinates

$$\Gamma(d) = \text{Re}(\Gamma) + j\text{Im}(\Gamma)$$

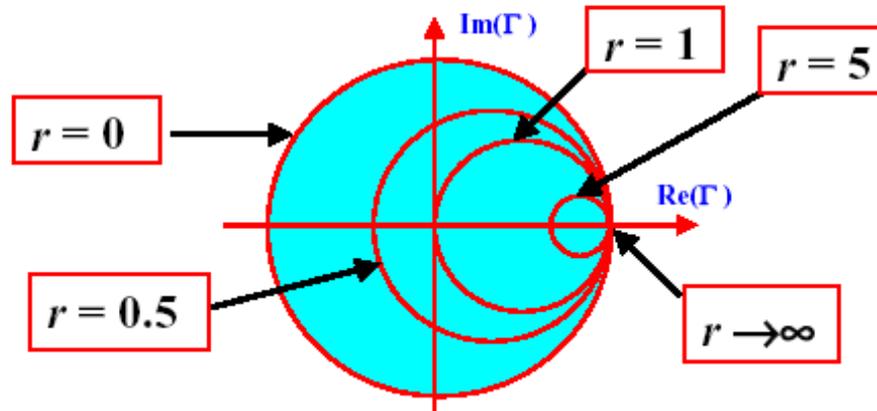
Now we can write

$$\begin{aligned} r + jx &= \frac{1 + \text{Re}(\Gamma) + j\text{Im}(\Gamma)}{1 - \text{Re}(\Gamma) - j\text{Im}(\Gamma)} \\ &= \frac{1 - \text{Re}^2(\Gamma) - \text{Im}^2(\Gamma) + j2\text{Im}(\Gamma)}{(1 - \text{Re}(\Gamma))^2 + \text{Im}^2(\Gamma)} \end{aligned}$$

The result for the **real part** indicates that on the complex plane with coordinates $(\text{Re}(\Gamma), \text{Im}(\Gamma))$ all the possible impedances with a given normalized resistance r are found on a **circle** with

$$\text{Center} = \left\{ \frac{r}{1+r}, 0 \right\} \qquad \text{Radius} = \frac{1}{1+r}$$

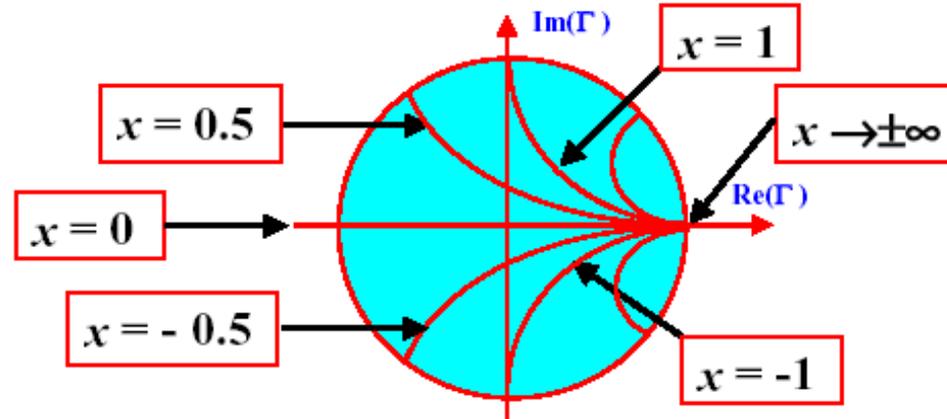
As the normalized resistance r varies from 0 to ∞ , we obtain a family of circles completely contained inside the domain of the reflection coefficient $|\Gamma| \leq 1$.



The result for the **imaginary part** indicates that on the complex plane with coordinates $(\text{Re}(\Gamma), \text{Im}(\Gamma))$ all the possible impedances with a given normalized reactance x are found on a **circle** with

$$\text{Center} = \left\{ 1, \frac{1}{x} \right\} \qquad \text{Radius} = \frac{1}{x}$$

As the normalized reactance x varies from $-\infty$ to ∞ , we obtain a family of arcs contained inside the domain of the reflection coefficient $|\Gamma| \leq 1$.



Basic Smith Chart techniques for loss-less transmission lines

- Given $Z(d) \Rightarrow$ Find $\Gamma(d)$
 Given $\Gamma(d) \Rightarrow$ Find $Z(d)$
- Given Γ_R and $Z_R \Rightarrow$ Find $\Gamma(d)$ and $Z(d)$
 Given $\Gamma(d)$ and $Z(d) \Rightarrow$ Find Γ_R and Z_R
- Find d_{\max} and d_{\min} (maximum and minimum locations for the voltage standing wave pattern)
- Find the Voltage Standing Wave Ratio (VSWR)
- Given $Z(d) \Rightarrow$ Find $Y(d)$
 Given $Y(d) \Rightarrow$ Find $Z(d)$

Given Γ_R and $Z_R \Rightarrow$ Find the Voltage Standing Wave Ratio (VSWR)

The Voltage standing Wave Ratio or **VSWR** is defined as

$$VSWR = \frac{V_{\max}}{V_{\min}} = \frac{1 + |\Gamma_R|}{1 - |\Gamma_R|}$$

The **normalized impedance** at a **maximum location** of the standing wave pattern is given by

$$z(d_{\max}) = \frac{1 + \Gamma(d_{\max})}{1 - \Gamma(d_{\max})} = \frac{1 + |\Gamma_R|}{1 - |\Gamma_R|} = VSWR!!!$$

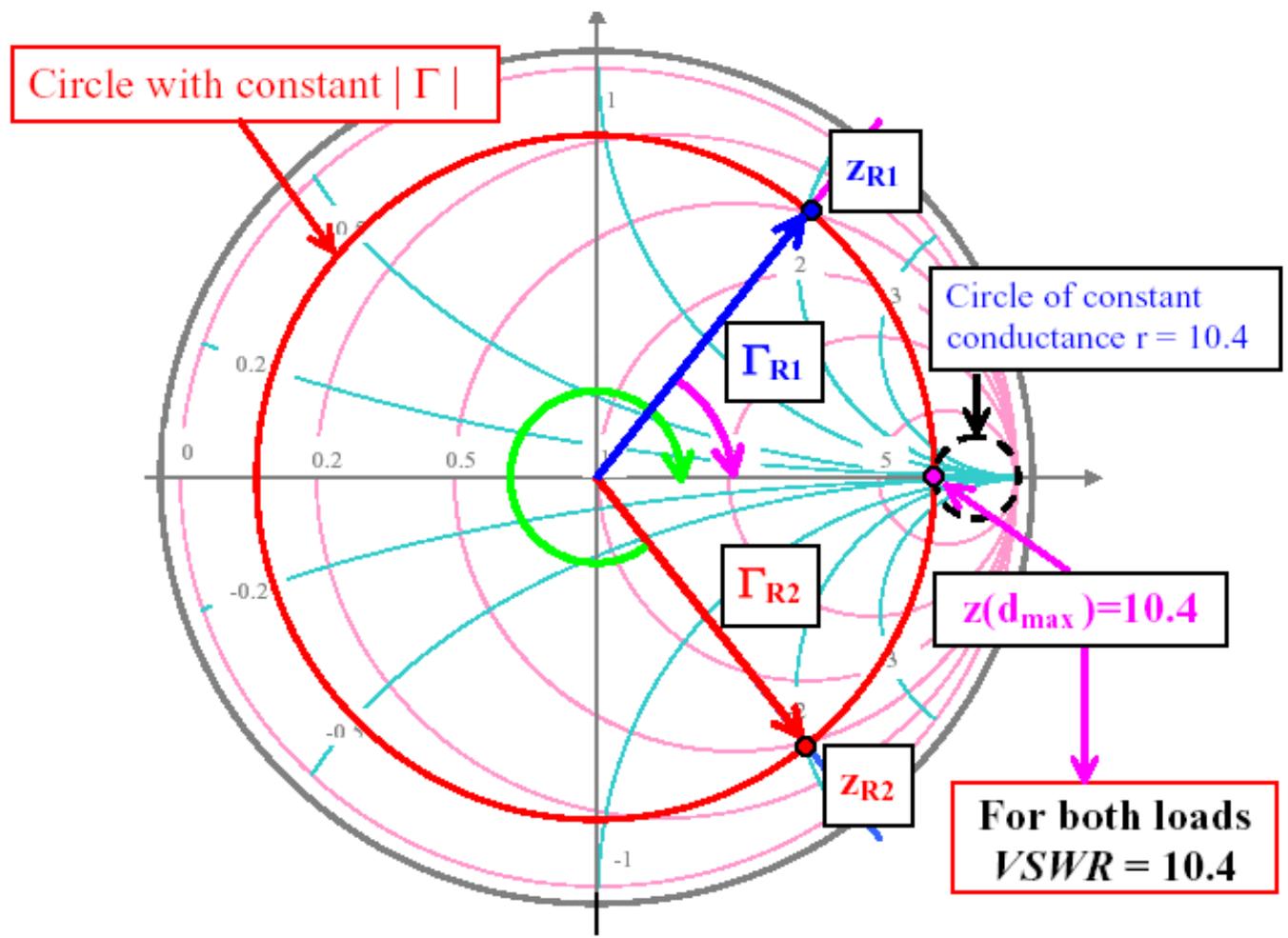
This quantity is always **real** and ≥ 1 . The VSWR is simply obtained on the Smith chart, by reading the value of the (real) normalized impedance, at the location d_{\max} where Γ is **real** and **positive**.

The graphical step-by-step procedure is:

1. Identify the load reflection coefficient Γ_R and the normalized load impedance Z_R on the Smith chart.
2. Draw the circle of constant reflection coefficient amplitude $|\Gamma(d)| = |\Gamma_R|$.
3. Find the intersection of this circle with the real positive axis for the reflection coefficient (corresponding to the transmission line location d_{\max}).
4. A circle of **constant normalized resistance** will also intersect this point. Read or interpolate the value of the normalized resistance to determine the *VSWR*.

Example: Find the *VSWR* for

$$Z_{R1} = 25 + j100 \Omega ; Z_{R2} = 25 - j100 \Omega \quad (Z_0 = 50 \Omega)$$



Given $Z(d) \iff$ Find $Y(d)$

Note: The normalized impedance and admittance are defined as

$$z(d) = \frac{1 + \Gamma(d)}{1 - \Gamma(d)} \quad y(d) = \frac{1 - \Gamma(d)}{1 + \Gamma(d)}$$

Since

$$\Gamma\left(d + \frac{\lambda}{4}\right) = -\Gamma(d)$$

$$\Rightarrow z\left(d + \frac{\lambda}{4}\right) = \frac{1 + \Gamma\left(d + \frac{\lambda}{4}\right)}{1 - \Gamma\left(d + \frac{\lambda}{4}\right)} = \frac{1 - \Gamma(d)}{1 + \Gamma(d)} = y(d)$$

Keep in mind that the equality

$$z\left(d + \frac{\lambda}{4}\right) = y(d)$$

is only valid for normalized impedance and admittance. The **actual** values are given by

$$Z\left(d + \frac{\lambda}{4}\right) = Z_0 \cdot z\left(d + \frac{\lambda}{4}\right)$$

$$Y(d) = Y_0 \cdot y(d) = \frac{y(d)}{Z_0}$$

where $Y_0 = 1/Z_0$ is the characteristic admittance of the transmission

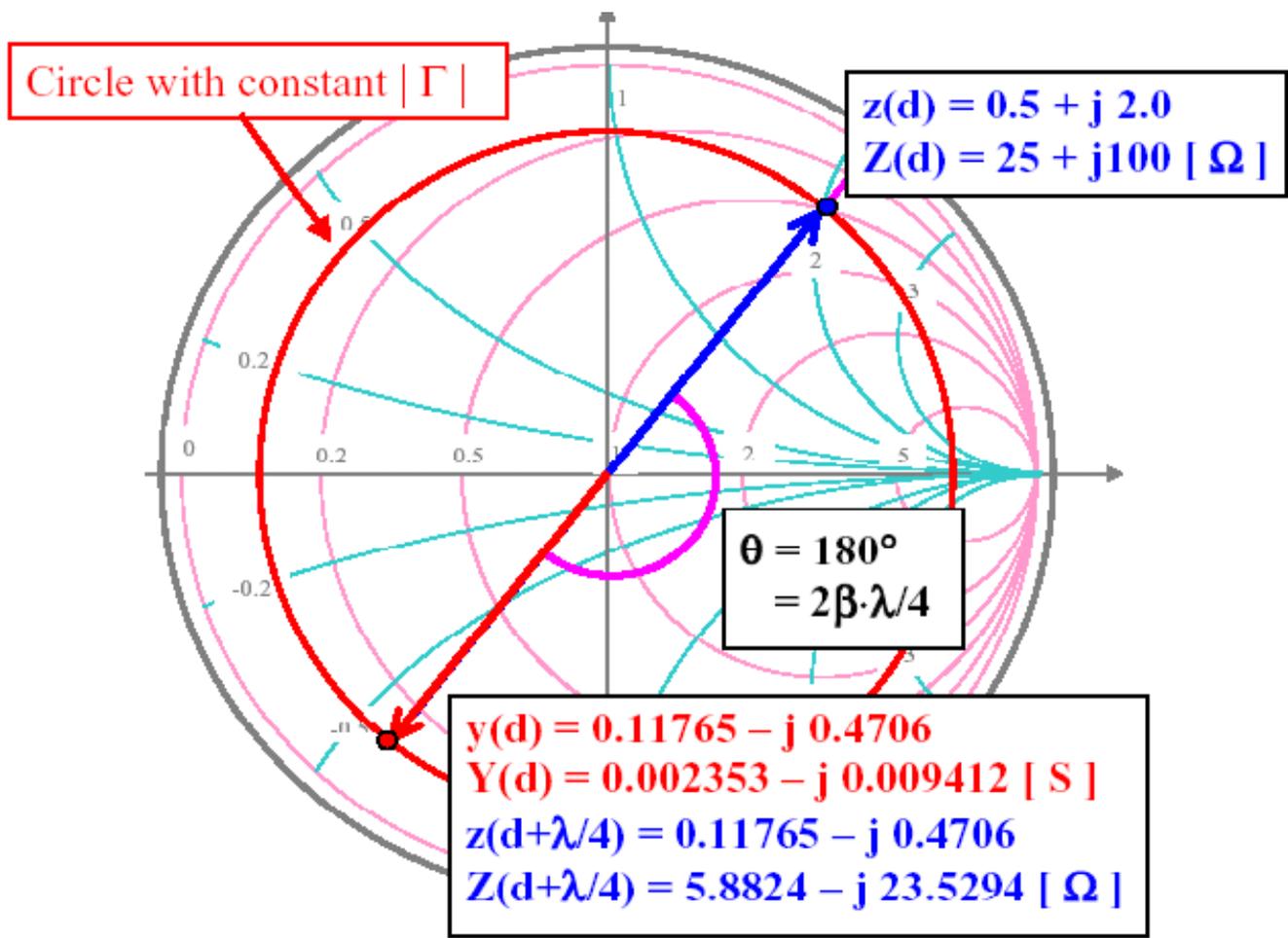
The graphical step-by-step procedure is:

1. Identify the load reflection coefficient Γ_R and the normalized load impedance Z_R on the Smith chart.
2. Draw the circle of constant reflection coefficient amplitude $|\Gamma(d)| = |\Gamma_R|$.
3. The **normalized admittance** is located at a point on the circle of constant $|\Gamma|$ which is diametrically opposite to the **normalized impedance**.

Example: Given

$$Z_R = 25 + j 100 \Omega \quad \text{with} \quad Z_0 = 50 \Omega$$

find Y_R .



Since related **impedance** and **admittance** are on opposite sides of the same Smith chart, the imaginary parts always have different sign.

Therefore, a **positive (inductive) reactance** corresponds to a **negative (inductive) susceptance**, while a **negative (capacitive) reactance** corresponds to a **positive (capacitive) susceptance**.

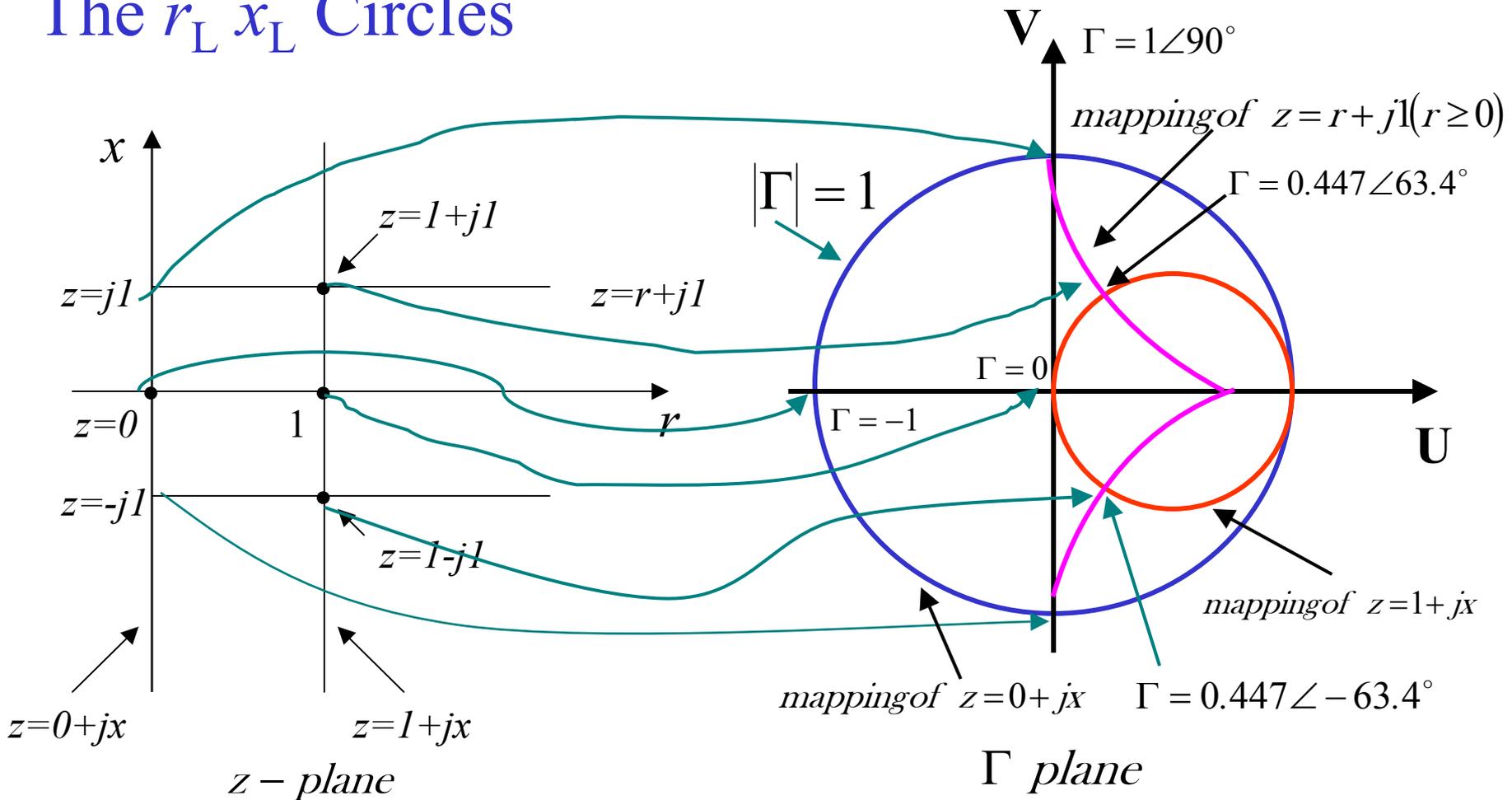
Numerically, we have

$$z = r + jx \quad y = g + jb = \frac{1}{r + jx}$$

$$y = \frac{r - jx}{(r + jx)(r - jx)} = \frac{r - jx}{r^2 + x^2}$$

$$\Rightarrow \quad g = \frac{r}{r^2 + x^2} \quad b = -\frac{x}{r^2 + x^2}$$

The $r_L x_L$ Circles



$$\left(z = \frac{Z}{Z_0} = \frac{R + jX}{Z_0} = r + jx \right)$$

Γ plane
 $(\Gamma = U + jV)$

Features of the Smith Chart

- All resistance circles have centers on the horizontal $\Gamma_i=0$ axis and all pass through the point $\Gamma=1$.
- The reactance circles have centers on the vertical $\Gamma_r=1$ line, and pass through the point $\Gamma=1$.
- The resistance and reactance circles are orthogonal.
- Smith Chart can be used to compute the normalized input impedance at a distance l away from the load.
 - Plot the reflection coefficient (Γ) at the load
 - Rotate the point cw an amount $2\beta l$ around the center of the chart (toward the generator)
- One full rotation around the center of S.C. corresponding to a phase shift of 0.5λ . A 180° rotation corresponding to $\lambda/4$ transformation; this facilitate the impedance circles to be used as admittance circles.

Features of the Smith Chart

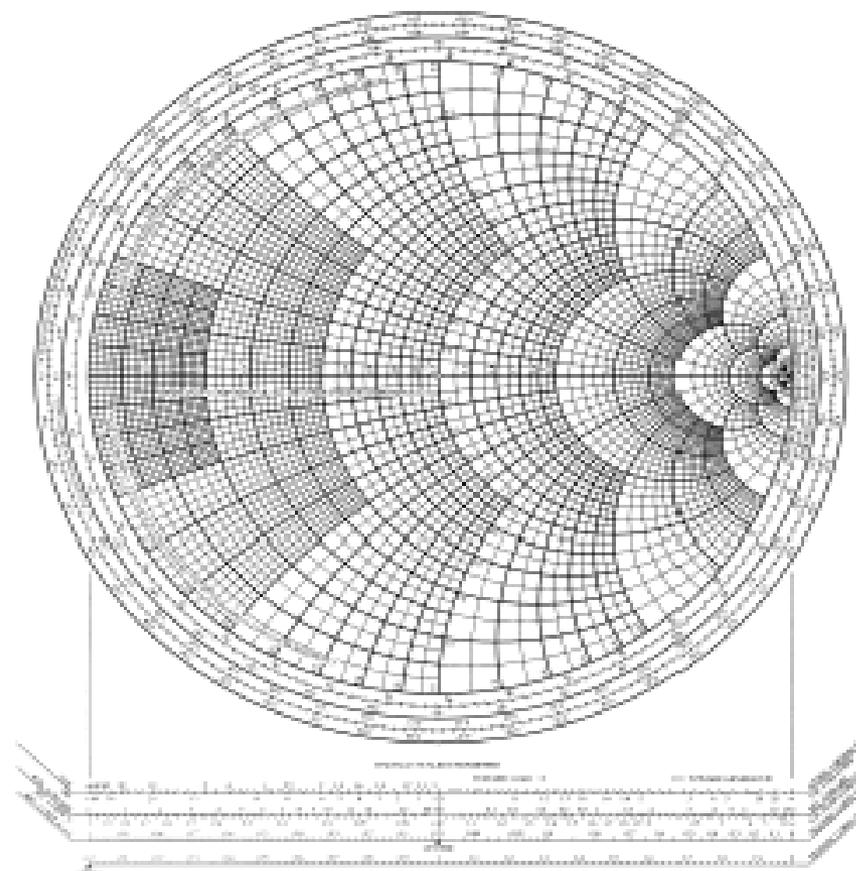
- Γ plotted directly in magnitude and phase
- Outside of chart ($r = 0$ circle) is $\Gamma = 1$
- Center of chart is $\Gamma = 0$ (i.e.: $Z = Z_0$)
- Top half – inductive x or capacitive b circles
- Bottom half – capacitive x or inductive b circles
- Left side is short – right side is open
- Position on TL? – Rotation, λ toward generator or load

The Z Smith Chart

- Only impedance circles are plotted on the Γ -plane.
- The chart looks quite “clean” but conversion between Y and Z must be done by a $\lambda/4$ rotation.

| | | |
|-------------------------------------|--|-----------|
| NAME | TITLE | ORIG. NO. |
| (SMITH CHART FORM 39-B (REV. 3-58)) | RAY ELECTRIC COMPANY, PINE BROOK, N.J. 07641 PRINTED IN U.S.A. | DATE |

IMPEDANCE OR ADMITTANCE COORDINATES

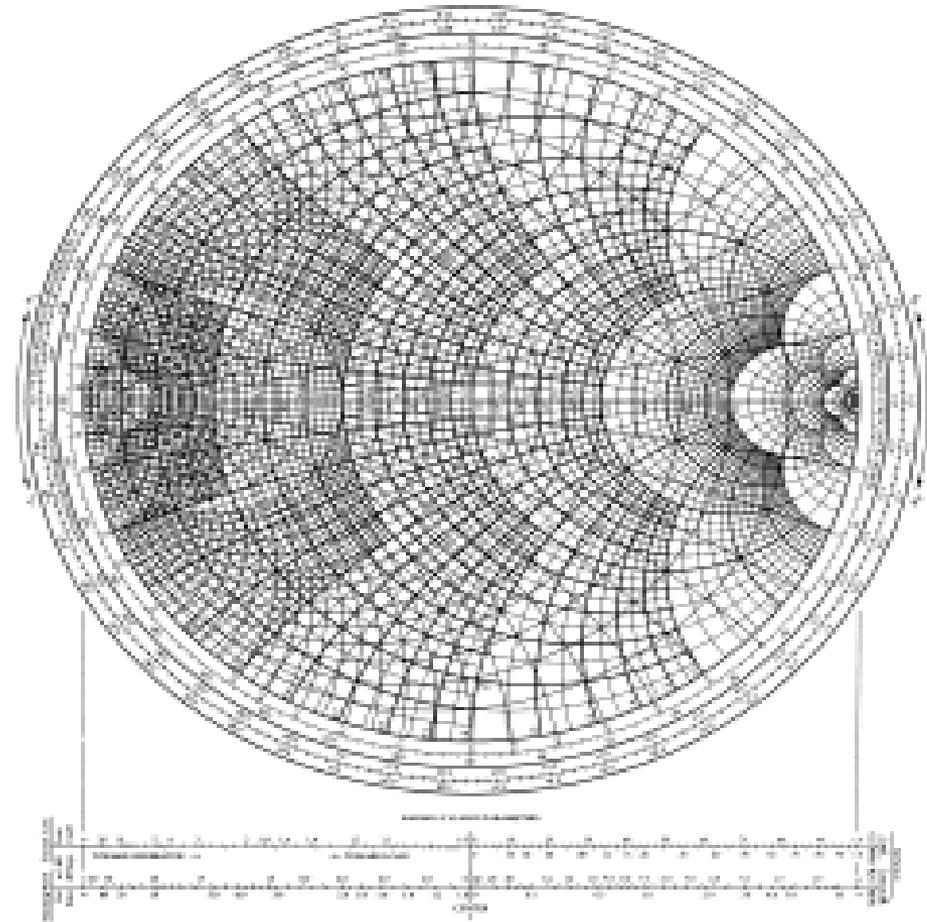


The ZY Smith Chart

- Impedance and admittance circles are both plotted on the Γ -plane.
- The chart looks confusing but conversion between Y and z is automatic.

| NAME | FILE | SWR. NO. |
|---------------------------|--|----------|
| SMITH CHART FORM 07-11-64 | ANALOG INSTRUMENTS COMPANY, NEW PROVIDENCE, N.J. 07094 | DATE |

NORMALIZED IMPEDANCE AND ADMITTANCE COORDINATES



standing waves result when a voltage generator of output voltage ($V_G = 1\sin\omega t$) and source impedance Z_G drive a load impedance Z_L through a transmission line having characteristic impedance Z_0 , where $Z_G = Z_0/Z_L$ and where angular frequency ω corresponds to wavelength l (b). The values shown in Figure a result from a reflection coefficient of 0.5.

Here, probe A is located at a point at which peak voltage magnitude is greatest—the peak equals the 1-V peak of the generator output, or incident voltage, plus the in-phase peak reflected voltage of 0.5 V, so on your oscilloscope you would see a time-varying sine wave of 1.5-V peak amplitude (trace c). At point C, however, which is located one-quarter of a wavelength ($l/4$) closer to the load, the reflected voltage is 180° out of phase with the incident voltage and subtracts from the incident voltage, so peak magnitude is the 1-V incident voltage minus the 0.5-V reflected voltage, or 0.5 V, and you would see the red trace. At intermediate points, you'll see peak values between 0.5 and 1.5 V; at B (offset $l/8$ from the first peak) in c, for example, you'll find a peak magnitude of 1 V. Note that the standing wave repeats every half wavelength ($l/2$) along the transmission line. The ratio of the maximum to minimum values of peak voltage amplitude measured along a standing wave is the standing wave ratio, SWR, $SWR=3$.

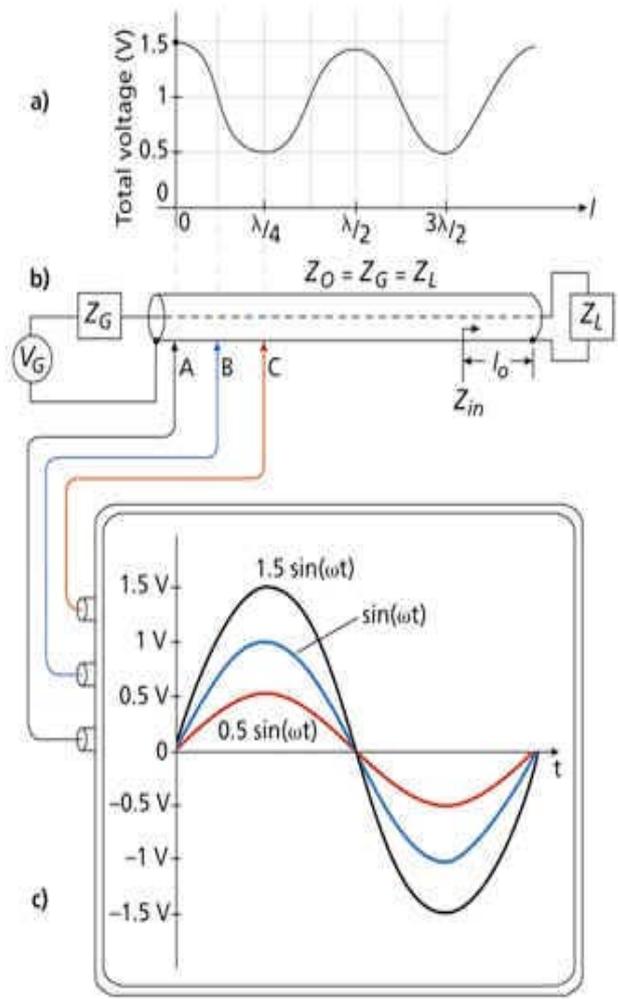
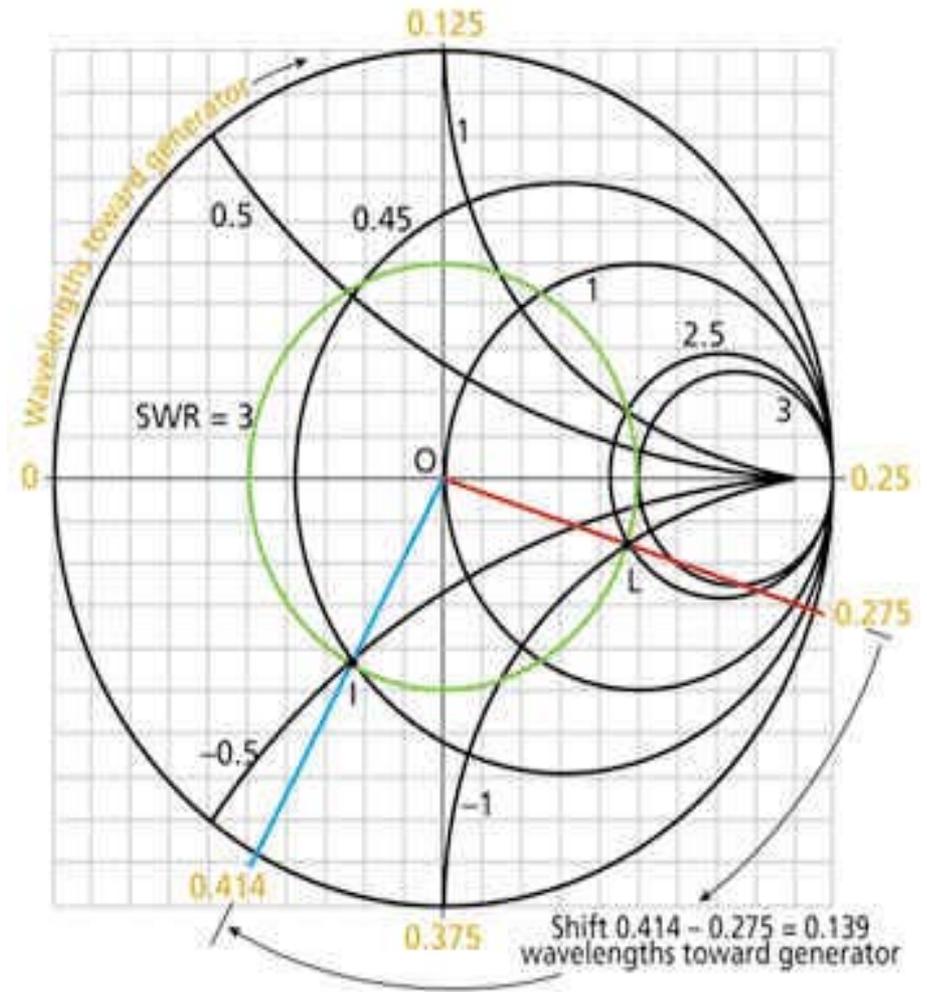


Figure 1

Here, point L represents a normalized load impedance $z_L = 2.5 - j1 = 0.5/18^\circ$ (I chose that particular angle primarily to avoid the need for you to interpolate between resistance and reactance circles to verify the results). The relationship of reflection coefficient and SWR depends only on the reflection coefficient magnitude and not on its phase. If point L corresponds to $|G| = 0.5$ and $SWR = 3$, then any point in the complex reflection-coefficient plane equidistant from the origin must also correspond to $|G| = 0.5$ and $SWR = 3$, and a circle centered at the origin and whose radius is the length of line segment OL represents a locus of constant-SWR points. (Note that the $SWR = 3$ circle here shares a tangent line with the $r_L = 3$ circle at the real axis; this relationship between SWR and r_L circles holds for all values of SWR.)



Using the standing-wave circle, you can determine input impedances looking into any portion of a transmission line such as Figure 1's if you know the load impedance. Figure 1, for instance, shows an input impedance Z_{in} to be measured at a distance l_0 from the load (toward the generator). Assume that the load impedance is as given by point L in Figure 2. Then, assume that l_0 is 0.139 wavelengths. (Again, I chose this value to avoid interpolation.) One trip around the Smith chart is equivalent to traversing one-half wavelength along a standing wave, and Smith charts often include 0- to 0.5-wavelength scales around their circumferences (usually lying outside the reflection-coefficient angle scale previously discussed).

Such a scale is shown in yellow in Figure 2, where clockwise movement corresponds to movement away from the load and toward the generator (some charts also include a counter-clockwise scale for movement toward the load).

Using that scale, you can rotate the red vector intersecting point L clockwise for 0.139 wavelengths, ending up at the blue vector. That vector intersects the SWR = 3 circle at point I, at which you can read Figure 1's input impedance Z_{in} . Point I lies at the intersection of the 0.45 resistance circle and -0.5 reactance circle, so $Z_{in} = 0.45 - j0.5$.

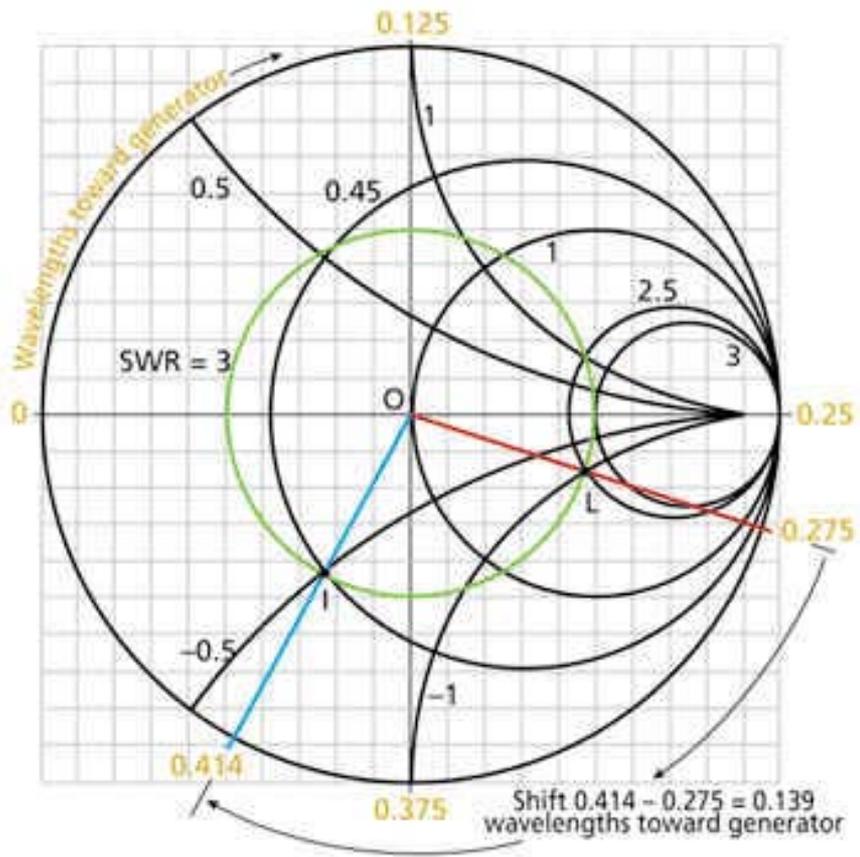
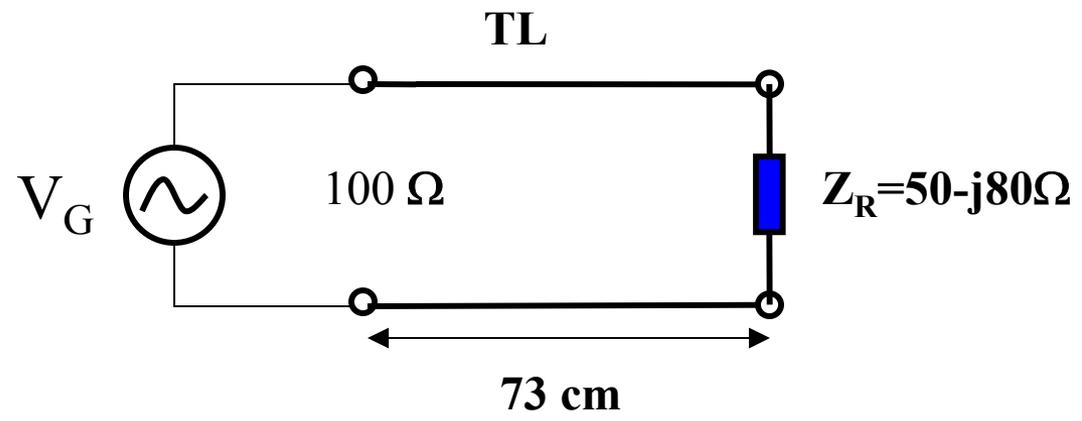


Figure 2

Example Problem:

A $100\text{-}\Omega$ transmission line with an air dielectric is terminated by a load $50\text{-}j80\Omega$. Determine the values of the VSWR, the reflection coefficient, and the percentage of the reflected power. If the power incident at the load is 100 mW , calculate the power in the load. If the generator frequency is 3 GHz and the line is 73 cm long, find the input impedance at the generator. What are the values of the maximum and minimum impedances existing on the line?



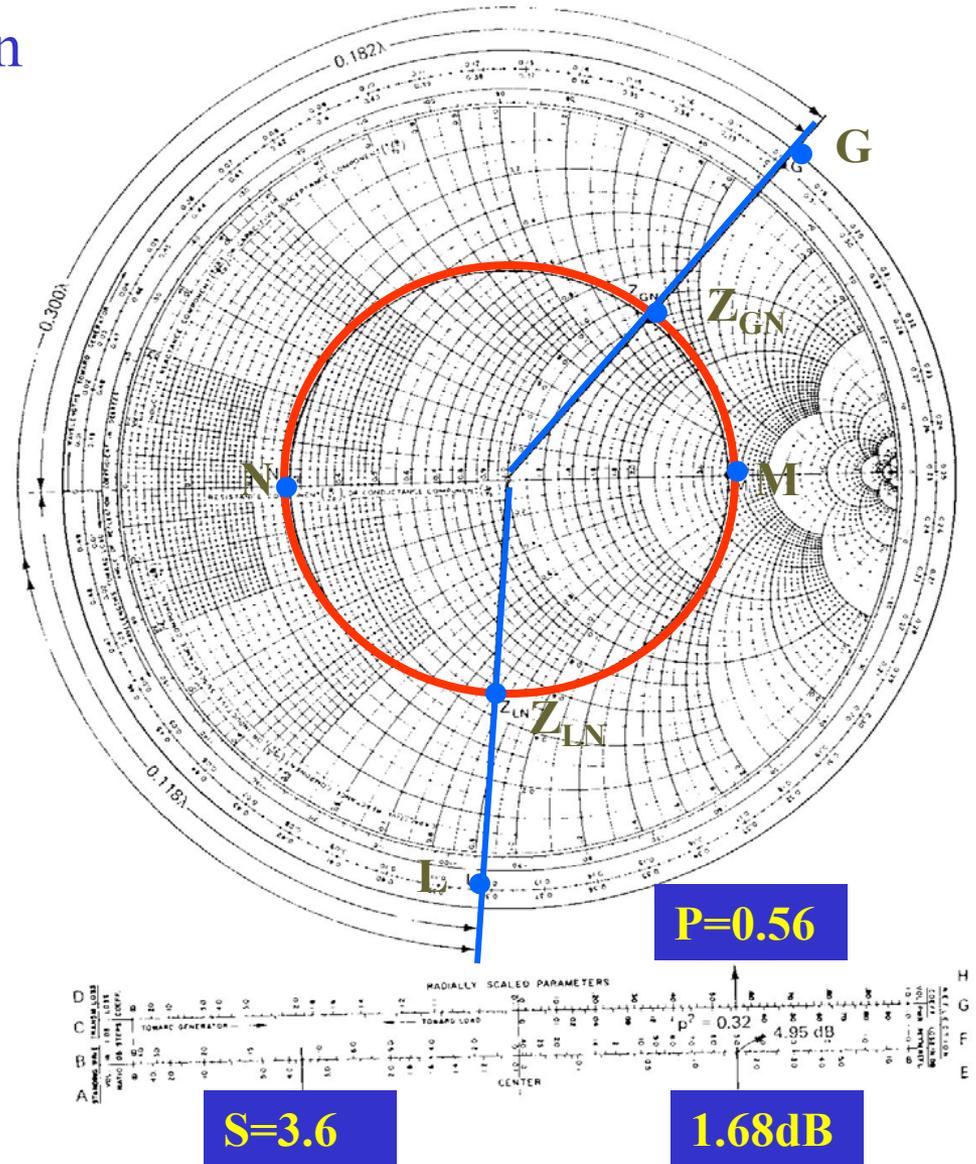
Solution

Step 1: Normalize Z_{LN}

$$Z_{LN} = \frac{50 - j80}{100} = 0.5 - j0.89$$

Step 2: Plot Z_{LN} and draw VSWR circle thru Z_{LN} .

Step 3: Mark off a length equal to the circles' radius on the H scale. The reflection coefficient is $P=0.56$. Same on the G scale for reflected power, $P^2 \times 100 = 0.32 \times 100 = 32\%$. The reflected power is $(32/100) \times 100 = 32$ mW and the power in the load is $100 - 32 = 68$ mW.



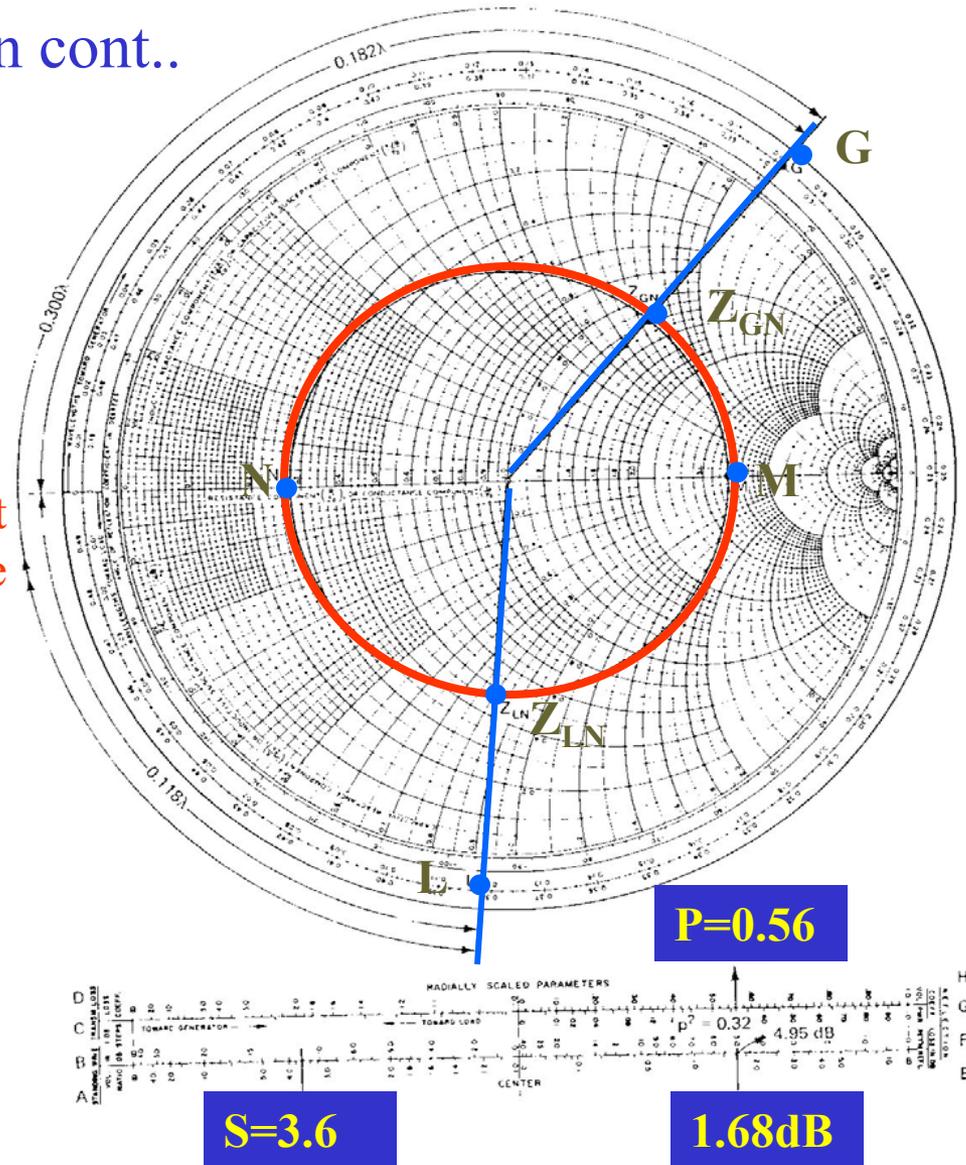
Solution cont..

Notice that the reflected power is $10 \log(32/100) = -4.95\text{dB}$ w.r.t the incident power while the load power when compared with the incident power is $10 \log(68/100) = -1.68\text{dB}$

Step 4: From the center of the chart draw a line thru Z_{LN} and extend the line to the peripheral scales.

$$\lambda = \frac{c}{f} = \frac{3 \times 10^{10} \text{ cm/sec}}{3 \times 10^9 / \text{sec}} = 10 \text{ cm}$$

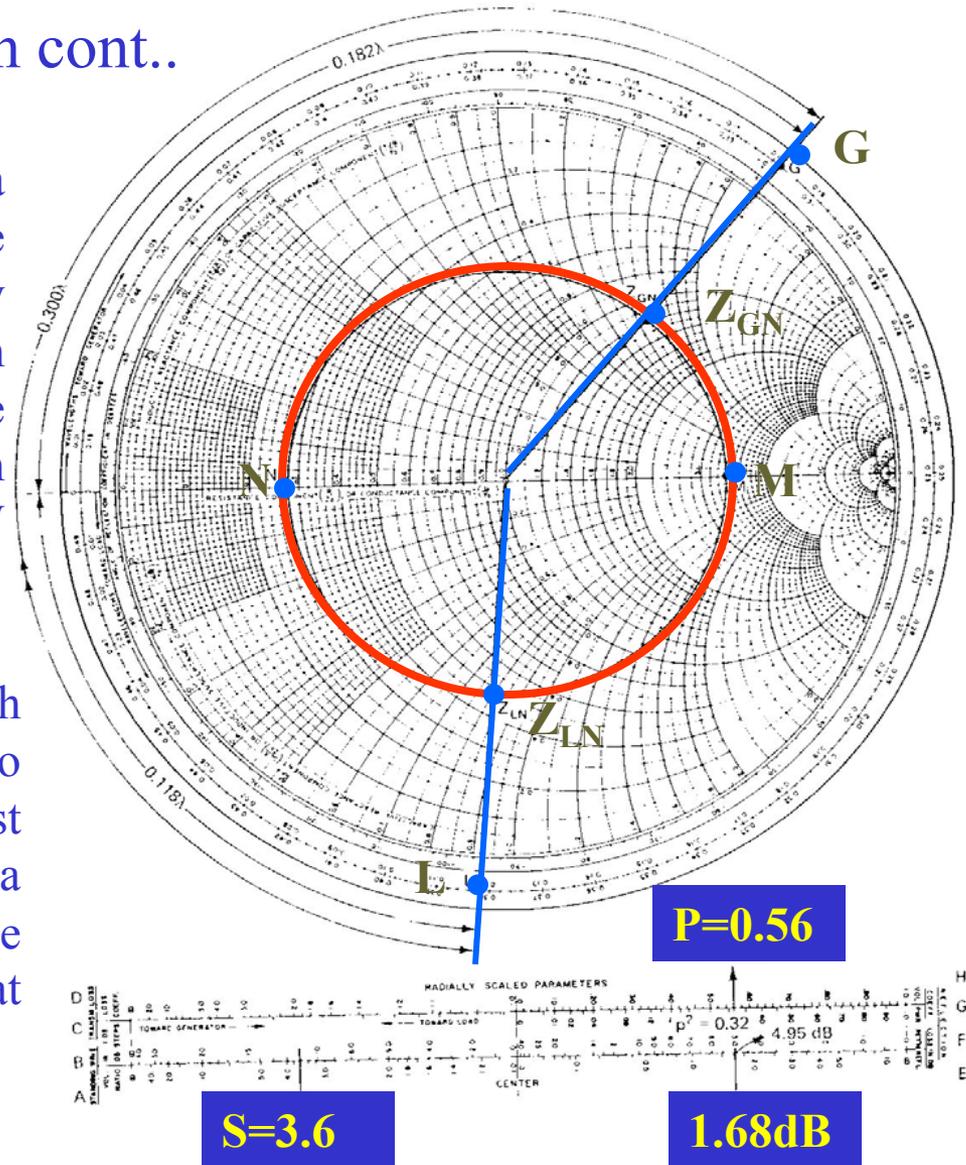
Therefore, 73 cm is equivalent to 7.3λ .



Solution cont..

Identical impedance values on a mismatched line repeat every half-wave length, which is the distance covered by one complete revolution on the Smith chart. A distance of 7.3λ will require 14 complete revolution together with additional rotation of 0.3λ in the CW direction toward the generator.

On the inner peripheral wavelength scale, the distance from Z_{LN} position to the null position, N, is 0.118λ . We must then travel toward the generator a further $0.3\lambda - 0.118\lambda = 0.182\lambda$ on the outmost peripheral scale and arrive at G.



Solution cont..

Step 5: Draw a line from G to the center of chart. Point Z_{GN} of intersection between this line and the VSWR circle represent the normalized input impedance at the generator; therefore, $Z_{GN} = 1.18 + j1.43$ and the de-normalized value is $118 + j143 \Omega$. The min. and max. impedance values on the line, respectively occur at N and M. At N the normalized value is $0.28 + j0$ so min. impedance is 28Ω (Z_0/S) and the M is $3.6 + j0$ so max. impedance is 360Ω or ($S \times Z_0$).

